SIMULTANEOUS ZERO SHIFTING AND ZERO RETAINING

IN A REACTANCE FUNCTION

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

In this research, the conditions for shifting some zeros of reactance functions to desired locations and at the same time leaving the remaining zeros in their original locations have been investigated. The reactance paramater z_{11} (or the susceptance parameter y_{22}) is chosen for convenience.

The necessary and sufficient conditions for the proposed zero shifting are the functions of the radian frequency values of original and desired zero locations, the radian frequency values of the poles, and the residues in the poles. As a prelude to finding the more general conditions, the proofs of the conditions are given for two cases of z_{11}, each of which has two pairs of purely imaginary zeros. A simple matrix equation is given to determine the component functions which are to be removed in the zero shifting process. This method is extended to include the general case of shifting i pairs of zeros to desired locations and leaving the others the same. Two applications of these methods to synthesize transfer functions by means of a pi or parallel ladder network are considered. These are a modified zero shifting synthesis and a parallel ladder synthesis.

CHAPTER I

INTRODUCTION

The problem of the synthesis of transfer functions with grounded two port networks is of considerable interest to network as well as control system designers. In particular, the pi network consisting of two kinds of elements is worthy of consideration, inasmuch as the synthesis problem has been solved in various ways.

A convenient method of synthesis for the ladder networks, within a constant multiplier, of a given transfer function is exemplified by a two-step zero shifting-zero producing technique (3,4,5)*. First, a set of parameters (for example, z_{11} and z_{12} or y_{22} and y_{12}) is determined from a given transfer function. These parameters must satisfy conditions for physical realizability (10). Second, z_{11}(or y_{22}) is synthesized as a driving point function having proper transmission zeros in series and shunt arms. The series and/or shunt elements are determined in the zero shifting-zero producing processes.

An area of investigation which has not been pursued actively heretofore is that of obtaining desired transfer characteristics by means of a pi or tee network instead of an extended ladder network. The synthesis of transfer functions with the pi or tee network requires that all zero shifting be conducted simultaneously in one step leaving

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* Numbers in parentheses denote entries in the List of References.

the remainder of the network with the desired transmission zeros. This research deals with this problem.

The objectives considered in the research are listed below.

(a) Formulate a method to shift one pair of reactance functions to the desired locations, leaving the others the same, and extend the method to include the shift of i zeros.

(b) Develop conditions for realizability by method of (a).

(c) Investigate methods and procedures for applying the results of (a) and (b) to the synthesis of transfer functions.

The transfer functions that will be considered are $K(s) = E_2/E_1$, with the network arrangement in Fig. 1. This is one of the most commonly used response-to-excitation transfer functions(7).

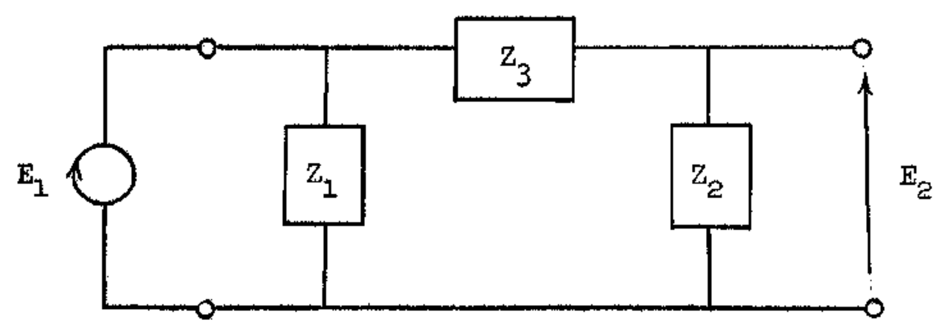

Fig. 1 Grounded Two-Port Pi(π) Network

This discussion of the transfer functions and the network parameters will be given in terms of LC-networks. However, it is known that by means of simple frequency transformations(10), the results in the LC-case can be made applicable to networks containing any two kinds of elements.

It is assumed that the realizability conditions of the reactance parameters and susceptance parameters are satisfied.

In the synthesis of a two-port network for a given transfer function, the related network parameters are found from established relations and then a network is realized having these parameters. For the network arrangement in Fig. 1, the transfer function used for the pi network is a voltage ratio described as

$$K(s) = \frac{E_2}{E_1} = \frac{P(s)}{Q(s)} = \frac{Z_2(s)}{Z_2(s) + Z_3(s)} \tag{1-1}$$

The susceptance parameters can be obtained by considering the network arrangement of Fig. 2 which is a dual of Fig. 1. The transfer function used for the tee network in Fig. 2 is a current ratio, that is

$$K'(s) = \frac{I_2}{I_1} = \frac{P'(s)}{Q'(s)} = \frac{Y_2(s)}{Y_2(s) + Y_3(s)} \tag{1-2}$$

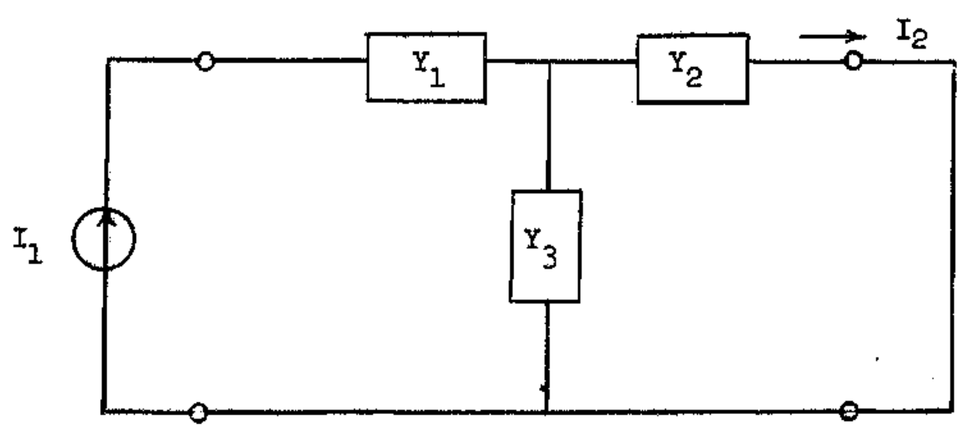

Fig. 2 Grounded Two-Port Tee(T) Network

The ratio, $P(s)/Q(s)$ or $P'(s)/Q'(s)$, is assumed to be in factored form and will be chosen to be

$$\frac{P(s)}{Q(s)} = \frac{P(s)/H(s)}{Q(s)/H(s)} \quad , \quad \frac{P'(s)}{Q'(s)} = \frac{P'(s)/H'(s)}{Q'(s)/H'(s)} \tag{1-3}$$

and

$$\begin{aligned} P(s) \text{ (or } P'(s) \text{)} &= A(s^2 + \omega_{\tau 1}^2)(s^2+\omega_{\tau 2}^2) \;\ldots\ldots(s^2+\omega_{\tau k}^2) \\ Q(s) \text{ (or } Q'(s) \text{)} &= (s^2+ \omega_{\beta 2}^2) \;\ldots\ldots\ldots\ldots(s^2+ \omega_{\beta n}^2) \end{aligned} \tag{1-4}$$

where A is a proportionality constant.

Since LC ladder networks are to be used, only those transfer functions with finite zeros and poles on the imaginary axis will be considered here. One method of synthesizing a given transfer function $K(s)$ in the form of (1-3) with an LC network arrangement as depicted in Fig. 1 is, first, to choose proper $H(s)$ such that both $P(s)/H(s)$ and $Q(s)/H(s)$ in (1-3) are in the form of realizable LC functions, secondly, to extract $Z_3(s)$ from $Q(s)/H(s)$ to obtain $Z_2(s)$ which has the form of $P(s)/H(s)$, and then realize $Z_2(s)$ by either Foster's or Cauer's reduction method. $H(s)$ chosen in this procedure must be an odd function and will be chosen to be

$$H(s) = s(s^2+ \omega_{\alpha 1}^2)(s^2+ \omega_{\alpha 2}^2)\ldots\ldots(s^2+ \omega_{\alpha m}^2) \tag{1-5}$$

Then, the parameters $Z_2(s)$ and $Z_3(s)$ can be identified as

$$Z_2(s) = \frac{P(s)}{H(s)} = A\,\frac{(s^2+ \omega_{\tau 1}^2)(s^2+ \omega_{\tau 2}^2)\ldots\ldots(s^2+ \omega_{\tau k}^2)}{s(s^2+ \omega_{\alpha 1}^2)(s^2+ \omega_{\alpha 2}^2)\ldots\ldots(s^2+ \omega_{\alpha m}^2)} \tag{1-6}$$

and

$$Z_2(s) + Z_3(s) = \frac{Q(s)}{H(s)} \qquad (1\text{-}7)$$

The equations (1-6) and (1-7) yield

$$Z_3(s) = \frac{Q(s)}{H(s)} - \frac{P(s)}{H(s)} \qquad (1\text{-}8)$$

The procedure depicted in (1-8) requires simultaneous zero shifting processes. It also requires simultaneous zero retaining processes, if $P(s)$ and $Q(s)$ have common terms. Therefore, $Z_3(s)$ corresponds to a zero shifting section, while $Z_2(s)$ corresponds to a zero producing section. In this sense, the zeros of $Z_2(s)$ can be termed "transmission zeros." By a procedure similar to that given for reactance parameters, the associated susceptance parameters are

$$Y_2(s) = \frac{P'(s)}{H'(s)} \qquad (1\text{-}9)$$

and

$$Y_3(s) = \frac{Q'(s) - P'(s)}{H'(s)} \qquad (1\text{-}10$$

The choice of $Z_3(s)$ (or $Y_3(s)$), as will be discussed in Chapter II and III, determines whether or not a particular network configuration is realizable. No attempt will be made to minimize the number of elements of synthesized networks.

CHAPTER II

A METHOD FOR SHIFTING ONE ZERO OF A REACTANCE TO A DESIRED LOCATION AND LEAVING THE OTHERS THE SAME

One way of specifying a typical reactance function (except for a constant multiplier) is shown in Fig. 3, which represents the zero-pole distribution on the positive real frequency axis.

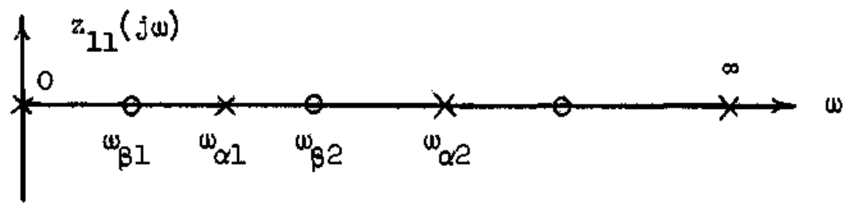

Fig. 3 Zero-Pole Distribution of Reactance Function

The zeros and poles are indicated by circles and crosses, respectively and these are denoted by the frequencies, $\omega_{\beta1}$, $\omega_{\alpha1}$, $\omega_{\beta2}$, $\omega_{\alpha2}$, ..., $\omega_{\beta n}$, $\omega_{\alpha m}$, ... The analytic expression for this function and transfer function, $z_{12}(s)$, can be written in factored form as follows.

$$z_{11}(s) = K\,\frac{(s^2+\omega_{\beta1}^2)(s^2+\omega_{\beta2}^2).....(s^2+\omega_{\beta n}^2)}{s(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2).....(s^2+\omega_{\alpha m}^2)} \tag{2-1}$$

$$z_{12}(s) = K'\,\frac{(s^2+\omega_{\tau1}^2)(s^2+\omega_{\tau2}^2)......(s^2+\omega_{\tau k}^2)}{s(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2)....(s^2+\omega_{\alpha m}^2)} \tag{2-2}$$

and

$$0 < \omega_{\beta 1} < \omega_{\alpha 1} < \omega_{\beta 2} < \omega_{\alpha 2} < \dots < \omega_{\alpha m} < \omega_{\beta n},$$

$$0 < \omega_{\tau 1} < \omega_{\tau 2} < \omega_{\tau 3} < \dots < \omega_{\tau k},$$

$$m = n - 1 \tag{2-3}$$

It is assumed that $z_{11}(s)$ and $z_{12}(s)$ have identical denominators and the poles of $z_{12}(s)$ are the poles of $z_{11}(s)$. It is also assumed that critical frequencies of zero and infinity are poles of $z_{11}(s)$ and $z_{12}(s)$. The admittance functions are also chosen to have the same forms as the impedance functions and are written

$$y_{22}(s) = H \frac{(s^2+\omega'^2_{\beta 1})(s^2+\omega'^2_{\beta 2})\dots\dots(s^2+\omega'^2_{\beta n})}{s(s^2+\omega'^2_{\alpha 1})(s^2+\omega'^2_{\alpha 2})\dots\dots(s^2+\omega'^2_{\alpha m})} \tag{2-4}$$

$$y_{12}(s) = H' \frac{(s^2+\omega'^2_{\tau 1})(s^2+\omega'^2_{\tau 2})\dots\dots(s^2+\omega'^2_{\tau k})}{s(s^2+\omega'^2_{\alpha 1})(s^2+\omega'^2_{\alpha 2})\dots\dots(s^2+\omega'^2_{\alpha m})} \tag{2-5}$$

and

$$0 < \omega'_{\beta 1} < \omega'_{\alpha 1} < \omega'_{\beta 2} < \omega'_{\alpha 2} < \dots\dots < \omega'_{\alpha m} < \omega'_{\beta n},$$

$$0 < \omega'_{\tau 1} < \omega'_{\tau 2} < \omega'_{\tau 3} < \dots\dots\dots < \omega'_{\tau k},$$

$$m = n - 1 \tag{2-6}$$

If the chosen parameters are not in the assumed forms, a simple transformation is required to make them in proper form. There are three other cases that need to be considered;

(a) When $z_{11}(s)$ has an odd numerator of one degree higher than the denominator, remove complete residues of $z_{11}(s)$ about infinity and take the reciprocal of remaining $z_{11}(s)$. The resultant function turns out to be in proper admittance form.

(b) When $z_{11}(s)$ has an odd numerator of one degree lower than the denominator, take the reciprocal of $z_{11}(s)$. This function becomes an admittance function in proper form.

(c) When $z_{11}(s)$ has an even numerator of one degree lower than the denominator, the removal of complete residues in any finite pole of $z_{11}(s)$ gives the remaining function in proper form.

The partial fraction expansions of $z_{11}(s)$ and $y_{22}(s)$ have the forms

$$z_{11}(s) = k_n s + \frac{k_o}{s} + \frac{k_1 s}{s^2 + \omega_{\alpha 1}^2} + \frac{k_2 s}{s^2 + \omega_{\alpha 2}^2} + \ldots\ldots\ldots\ldots \quad (2\text{-}7)$$

$$y_{22}(s) = h_n s + \frac{h_o}{s} + \frac{h_1 s}{s^2 + \omega_{\alpha 1}'^2} + \frac{h_2 s}{s^2 + \omega_{\alpha 2}'^2} + \ldots\ldots\ldots\ldots \quad (2\text{-}8)$$

where, k_n, k_o, k_1, ... and h_n, h_o, h_1, ... are all positive, real numbers.

2.1 <u>Conditions for Realizability by Zero Shifting Techniques with Given Network Configurations</u>

The reactance function and susceptance function in (2-1) and (2-4) can be represented as in (2-9) and (2-10), and their plots on the positive frequency axis may be shown in Fig. 4.

$$z_{11}(j\omega) = j X_1(\omega) \quad (2\text{-}9)$$

$$y_{22}(j\,\omega) = j\,B_1(\omega) \qquad (2\text{-}10)$$

where $X_1(\omega)$ and $B_1(\omega)$ are real values which are evaluated on the positive frequencies, ω.

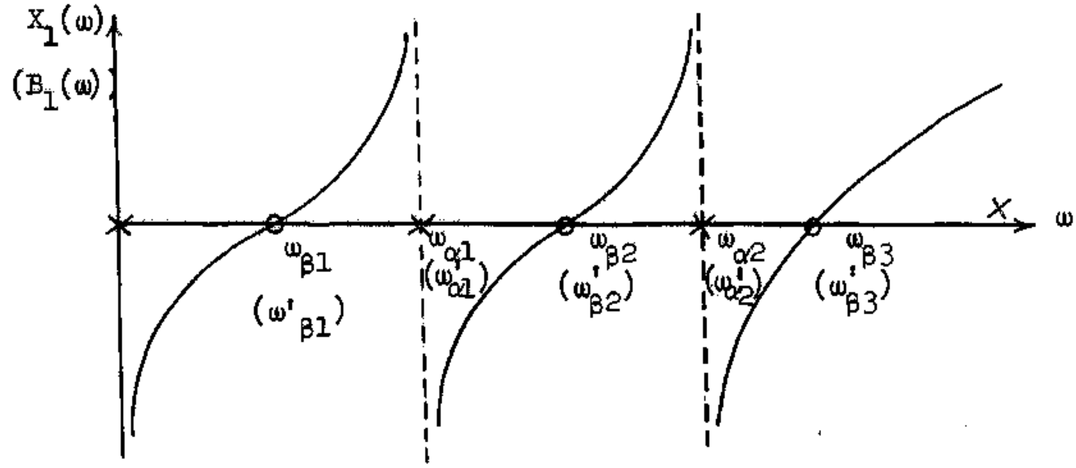

Fig. 4 Plot of Reactance Function versus Positive Frequency

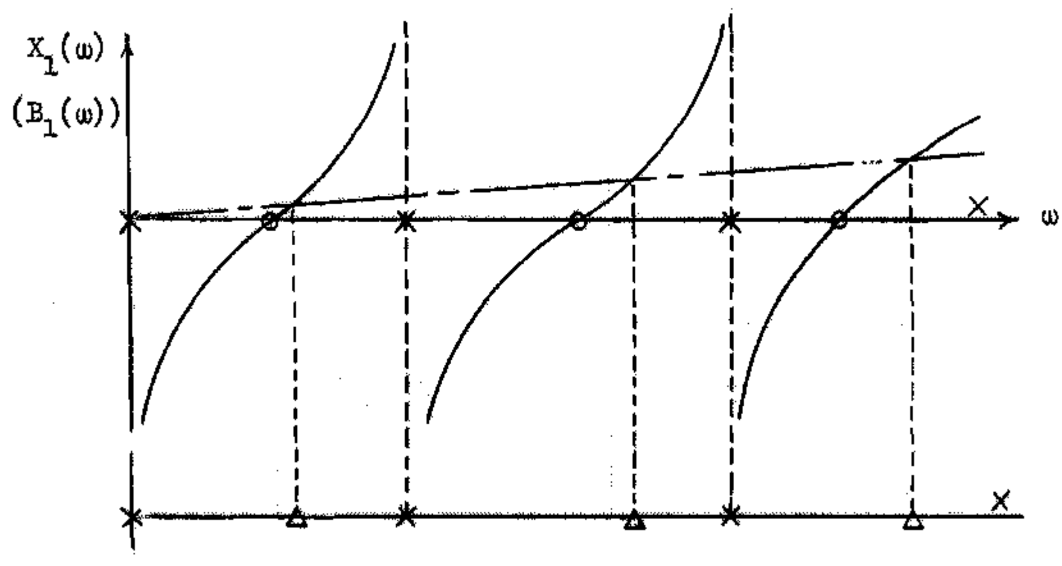

Fig. 5 Graphical Illustration of Zero Shifting Technique #1

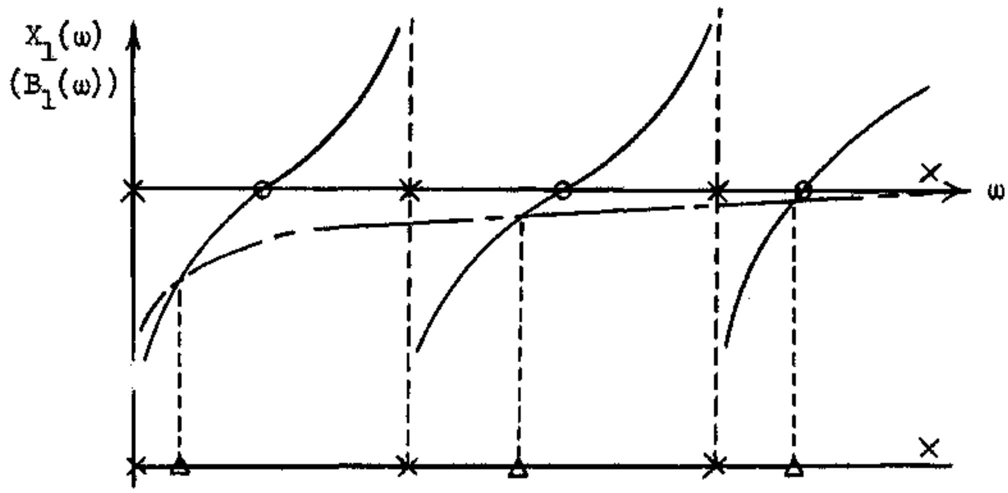

Fig. 6 Graphical Illustration of Zero Shifting Technique #2

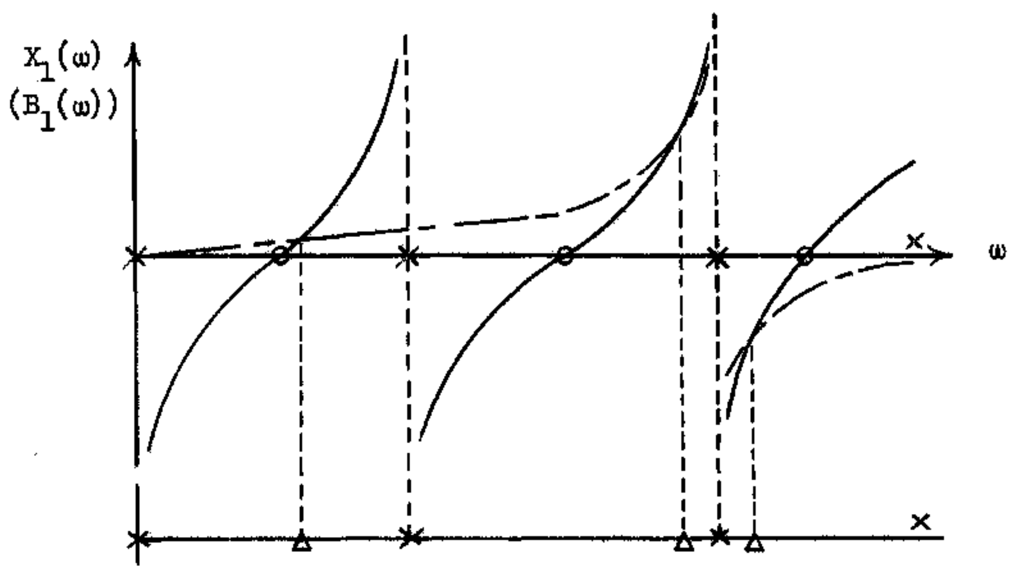

Fig. 7 Graphical Illustration of Zero Shifting Technique #3

Consider now the problem of zero shifting techniques (3,7,8). There are altogether three sets of zero shifting techniques which are summarized in Fig. 5, 6 and 7. These are called "zero shifting technique #1, #2 and #3." Fig. 5 shows the zero shifting technique by means of partial removal of the residue in the pole at $s = \infty$. In actual synthesis, a series inductance or shunt capacitance is used for the zero shifting technique #1, as shown in Fig. 8. These networks are termed "Type I sections" for convenience.

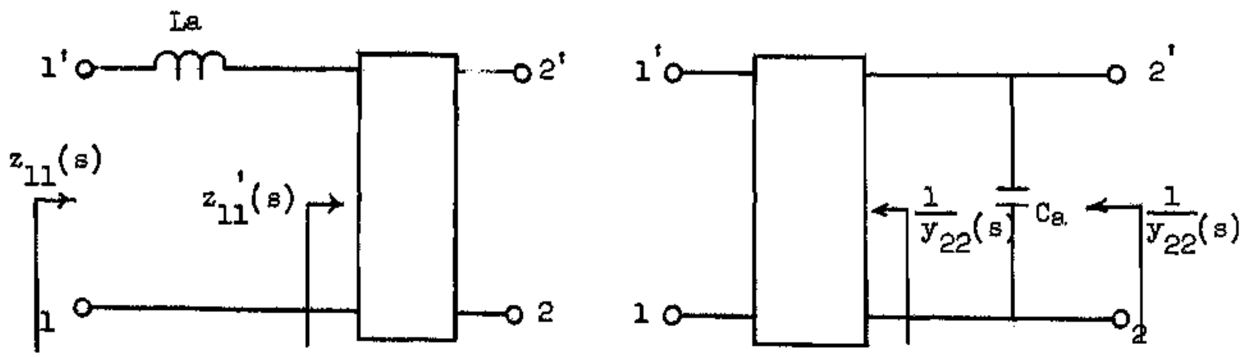

Fig. 8 Type I Sections

In the synthesis of a two port network with Type I sections, it is necessary that the sections be removed in a proper order. To establish the proper ordering, the minimum element value at transmission zeros must be removed first (6).

The minimum element value is defined as

$$L_a = \text{minimum}\ (z_{11}(j\,\omega_T)/\ j\,\omega_T\ ) \tag{2-11}$$

$$C_a = \text{minimum}\ (y_{22}(j\,\omega_T)/\ j\,\omega_T\ ) \tag{2-12}$$

Since only reactance parameters are considered, the instantaneous slope of $X_1(\omega)$ or $B_1(\omega)$ at any positive frequency, $\omega = \omega_1$, becomes

$$\left.\frac{\partial X_1(\omega)}{\partial\,\omega}\right|_{\omega = \omega_1} > 0 \tag{2-13}$$

$$\left.\frac{\partial B_1(\omega)}{\partial\omega}\right|_{\omega=\omega_1} > 0 \qquad (2\text{-}14)$$

But a zero shifting section which is made up of Type I sections has component functions with positive slopes as follow;

$$z_1(s) = d_1 s$$

$$\text{or} \quad z_1(j\omega) = jd_1\omega \qquad (2\text{-}15)$$

where d_1 is a positive, real number.

By extracting $z_1(j\omega)$ from $z_{11}(j\omega)$, the imaginary impedance level of remainder function, $z'_{11}(j\omega)$, is reduced and new zeros are shifted away from the origin. Therefore, when the frequency value of a transmission zero is less than that of the zero shifted, zero shifting technique #1 cannot be used. Hence, the condition 1:

<u>Condition 1</u>; Necessary conditions for realizability with Type I sections are

$$\text{for } z_{11}(s), \quad \omega_{\beta i} < \omega_\tau < \omega_{\alpha i} \quad i = 1,2,3\ldots$$

$$\text{for } y_{22}(s), \quad \omega'_{\beta i} < \omega'_\tau < \omega'_{\alpha i} \quad i = 1,2,3\ldots$$

where, ω_τ or ω'_τ is the frequency value of transmission zeros.

In Fig. 6 zero shifting is done by removing a partial residue in the pole at $s = 0$. This can be realized by the networks in Fig. 9, which will be called "Type II sections" for convenience.

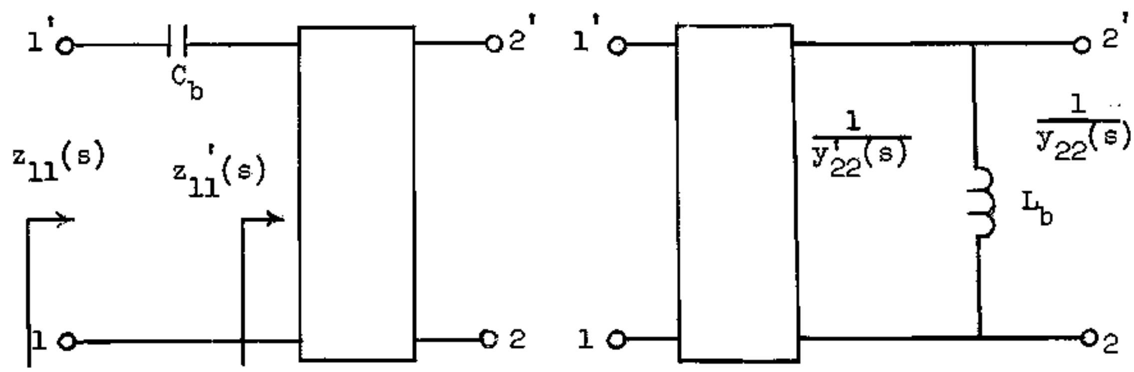

Fig. 9 Type II Sections

In Type II sections, zero shifting is achieved by a series capacitance or by a shunt inductance. The minimum element values in this case are defined as

$$C_b = \text{minimum } (j\,\omega_T \;\; z_{11}(j\,\omega_T)\,) \tag{2-16}$$

$$L_b = \text{minimum } (j\,\omega_T \;\; y_{22}(j\,\omega_T)\,) \tag{2-17}$$

In this case the zero shifting section has component functions of negative slopes.

$$z_1(s) = \frac{d_1}{s}$$

or

$$z_1(j\,\omega) = -j\,\frac{d_1}{\omega} \tag{2-18}$$

The remainder function, $z_{11}'(j\,\omega)$, which is obtained by extracting $z_1(j\,\omega)$ from $z_{11}(j\,\omega)$, has a higher imaginary impedance level than the original $z_{11}(j\,\omega)$. This results in all the zeros of $z_{11}(s)$ being shifted toward the origin. For this reason, zero shifting technique #2 is available only when the value of a transmission zero is less than the one of the original zero to be shifted. This fact is also applicable for the admittance parameters as well.

Hence, the condition 2:

Condition 2; Necessary conditions for realizability with Type II sections are

$$\text{for } z_{11}(s), \quad \omega_{\alpha i} < \omega_{\tau} < \omega_{\beta i+1} \quad , i = 1,2,3,\ldots\ldots$$

$$\text{for } y_{22}(s), \quad \omega'_{\alpha i} < \omega'_{\tau} < \omega'_{\beta i+1} \quad , i = 1,2,3,\ldots\ldots$$

where ω_τ or ω'_τ is the frequency value of transmission zeros. Zero shifting techniques by partial removal of finite pole residues described in Fig. 7 yield the resultant networks shown in Fig. 10. These networks will be called "Type III sections."

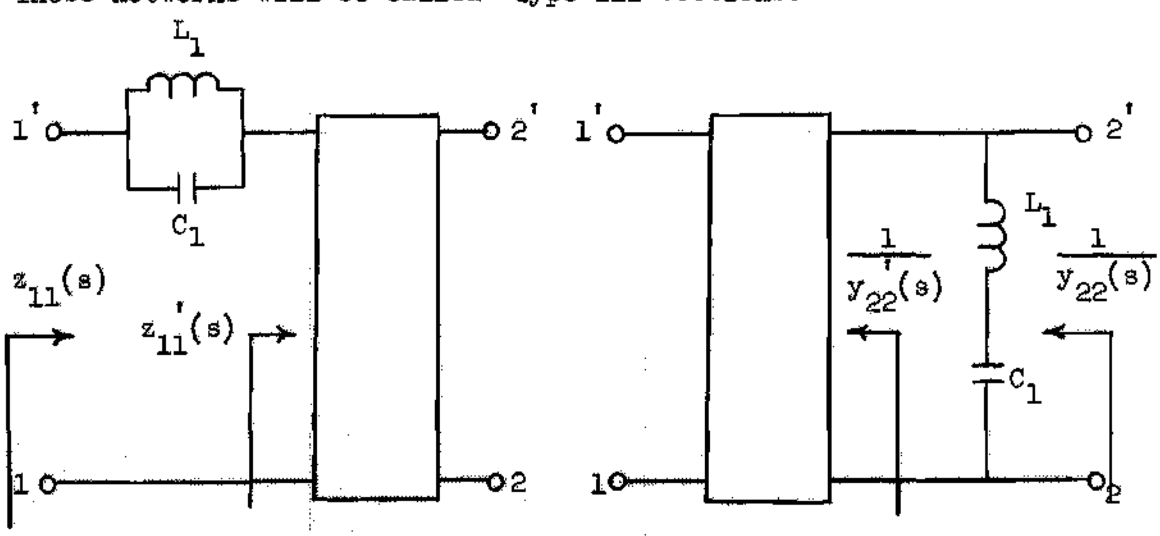

Fig. 10 Type III Sections

The component functions of zero shifting sections in this case become

$$z_1(s) = \frac{d_1 \; s}{s^2 + \omega_{\alpha i}^2}$$

or $$z_1(j\,\omega) = \frac{j d_1 \omega}{\omega_{\alpha i}^2 - \omega^2} \qquad (2\text{-}19)$$

Depending upon the choice of $\omega_{\alpha i}$, new zeros of $z_{11}'(s)$ are shifted toward or away from the origin. The $\omega_{\alpha i}$, which is the frequency value of the pole in a component function, is called the "control frequency." To choose appropriate control frequencies, two general rules must be obeyed.

Rule 1; For $X_1(\omega_T) > 0$, that is, $\omega_{\beta i} < \omega_T < \omega_{\alpha i}$, choose a higher adjoining pole for the control frequency. The higher adjoining pole is the pole at $s = j\,\omega_{\alpha i}$.

Rule 2; For $X_1(\omega_T)<0$, that is, $\omega_{\alpha i-1} < \omega_T < \omega_{\beta i}$, choose a lower adjoining pole for the control frequency. The lower adjoining pole is the pole at $s = j\,\omega_{\alpha i-1}$.

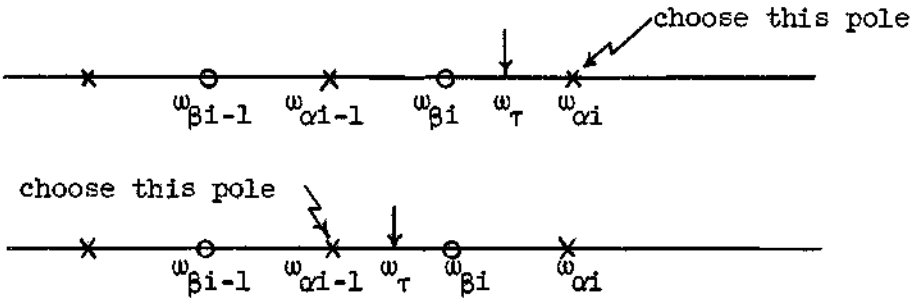

Fig. 11 Examples for Rule 1 and Rule 2

When $\omega_{\beta i} < \omega_{\tau} < \omega_{\alpha i}$ and $\omega_{\alpha i}$ is chosen for a control frequency, which is the case of Rule 1, the component function in (2-19) evaluated at $s = j\,\omega_{\beta i}$ has a positive imaginary value and the value of $z_{11}'(j\,\omega_{\beta i})$ will be negative, imaginary. Therefore, the zero at $s = j\,\omega_{\beta i}$ will be shifted away from the origin and the desired zero shifting is achieved. The converse statement also holds true.

When $\omega_{\alpha i-1} < \omega_{\tau} < \omega_{\beta i}$ and $\omega_{\alpha i-1}$ is chosen for a control frequency, the value of component function in (2-19) evaluated at $s = j\,\omega_{\beta i}$ is negative, imaginary and the zero at $s = j\,\omega_{\beta i}$ will be shifted toward the origin. This is the case of Rule 2. Therefore, both Rule 1 and Rule 2 are justified. A similar set of rules holds for the admittance function.

It is noted that there is no restriction on the location of transmission zeros when Type III sections are used for synthesizing a reactance function, provided the appropriate control frequency is chosen. The procedures for realizing a two port network with Type I, II and III sections can be found in the references (3,7,8).

2.2 A Method for Shifting One Zero Only

Description of Problem. In Art. 2.1, the zero shifting techniques imposed upon Type I, II and III sections were considered. In this section, the problem of shifting one zero of a reactance function to a proper location and leaving the others the same will be discussed.

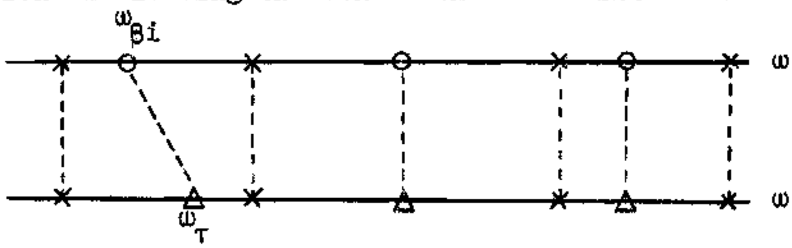

Fig. 12 Original and New Zero-Pole Distributions

Fig. 12 shows a set of zero pole distributions of a reactance function. In that figure original and new zeros are denoted by circles and triangles, respectively. These can be achieved by utilizing the combined zero shifting techniques discussed in Art. 2.1. Because of the similarity between impedance and admittance functions, only impedance function will be discussed here.

Procedure

Consider now the problem of shifting one zero alone along the positive frequency axis in 2-z, 3-z and n-z functions. The term "n-z function" means the reactance function which has n pairs of imaginary zeros, that is, n zeros on the positive frequency axis, and will be used throughout this paper. The network structure of Fig. 13 is made up on a predicted configuration of particular building blocks.

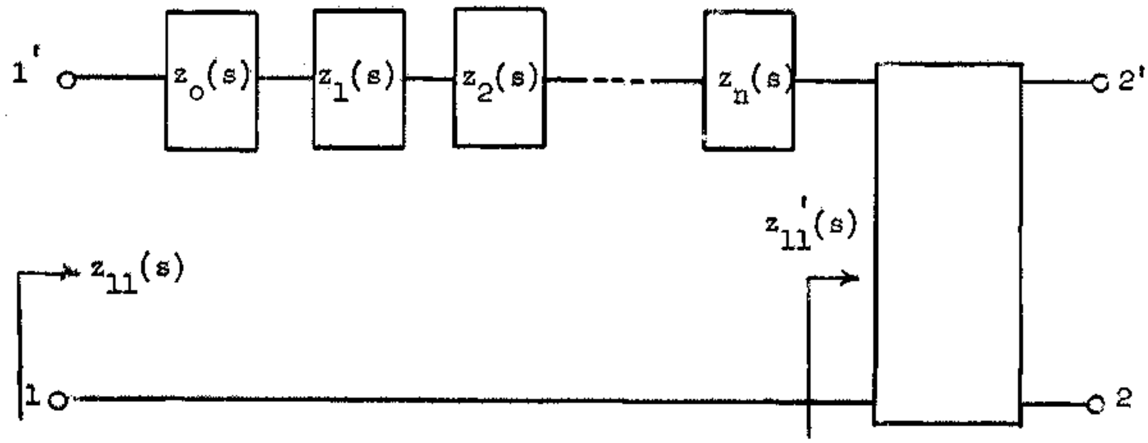

Fig. 13 Predicted Network Structure for Shifting One Zero Alone

The zero shifting section in Fig. 13 is composed of several component functions; $z_o(s)$, $z_1(s)$, $z_2(s)$,......and $z_n(s)$. These component functions can be grouped into three types; Type I-like, Type II-like and Type III-like sections. It is assumed that $z_n(s)$ and $z_o(s)$ are Type I-like and Type II-like sections and all the others ($z_1(s)$, $z_2(s)$,..., $z_{n-1}(s)$) are Type III-like sections. Then, the component functions of zero shifting section become

$$z_o(s) = \frac{d_o}{s}$$

$$z_i(s) = \frac{d_i\ s}{s^2 + \omega_{\alpha i}^2} \quad , i = 1,2,3,\ldots, n-1 \qquad (2\text{-}20)$$

$$z_n(s) = d_n s$$

The component functions must be properly chosen and arranged such that the remainder function $z_{11}'(s)$ has one zero at a desired location and all the other zeros at the same location as $z_{11}(s)$. Thus, the problem is reduced to determining d_o, d_i and d_n and to choosing proper $\omega_{\alpha i}$. For the successful synthesis of $z_{11}(s)$ by this procedure, the following two requirements must be met:

(a) d_o, d_i and d_n must be positive.

(b) $d_o < k_o$, $d_i < k_i$ and $d_n < k_n$ must be satisfied. (2-21)

k_o, k_i and k_n are taken from (2-7).

The first requirement shows that no negative elements are allowed in this process, while the second one comes from the fact that the remainder function must be positive real.

Procedure for 2-z functions

For a 2-z function, the impedance function $z_{11}(s)$ has the form

$$z_{11}(s) = K \frac{(s^2 + \omega_{\beta 1}^2)(s^2 + \omega_{\beta 2}^2)}{s(s^2 + \omega_{\alpha 1}^2)} \qquad (2\text{-}22)$$

There are three poles that are available for control frequencies; poles at $s = 0$, $s = j\,\omega_{\alpha 1}$, and $s = \infty$. There are also four situations for the location of ω_τ on the positive real frequency axis; $0 < \omega_\tau < \omega_{\beta 1}$, $\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1}$, $\omega_{\alpha 1} < \omega_\tau < \omega_{\beta 2}$, and $\omega_{\beta 2} < \omega_\tau < \infty$. Since the choice of control frequencies is different in each situation, they will be discussed separately.

Case 1; $0 < \omega_\tau < \omega_{\beta 1}$

From the proposed problem that the remainder function $z_{11}'(s)$ has zeros at $s = j\,\omega_\tau$ and $s = j\,\omega_{\beta 2}$, the following two relationships are known.

$$z_{11}'(j\,\omega_\tau) = 0 \tag{2-23}$$

$$z_{11}'(j\,\omega_{\beta 2}) = 0 \tag{2-24}$$

Since only two known conditions are available, at most two poles can be chosen for control frequencies. To choose appropriate control frequencies, the effect of extracting each possible component function must be examined. If a pole at s=0 is chosen for a control frequency in the zero shifting section, the zeros of $z_{11}(s)$ will be shifted toward the origin as in the zero shifting technique #2 in Art. 2.1. Because of the characteristics of Type I sections, the displacement of the zero at $s = j\,\omega_{\beta 2}$ is small compared to that of the zero at $s = j\,\omega_{\beta 1}$. If the displacement of the zero at $s = j\,\omega_{\beta 2}$ is compensated by the other component function, the proposed objective can be achieved. If a pole at $s = j\,\omega_{\alpha 1}$ is chosen for a component function, the zero at $s = j\,\omega_{\beta 1}$ will be shifted away from the origin and the zero at $s = j\,\omega_{\beta 2}$ will be shifted toward the origin as shown in Fig. 7. This requires two additional component functions to compensate the displacements from both the new zeros at $s = j\,\omega_\tau$ and $s = j\,\omega_{\beta 2}$. But only two component functions can be determined by two known conditions, and the pole at $s = j\,\omega_{\alpha 1}$ cannot be chosen for a component function. If a partial residue in the pole at $s = \infty$ is removed from original function $z_{11}(s)$, then all the zeros of $z_{11}(s)$ will be shifted away from the origin as depicted in Fig. 5. From the characteristics of Type I sections, the displacement of the zero at $s = j\,\omega_{\beta 1}$ is small compared to that of the zero at $s = j\,\omega_{\beta 2}$. This can be, therefore, utilized to compensate the zero disturbance resulting from removal of a partial residue in the pole at $s = 0$. From the above investigation, it is easily seen that the best choices of component functions are the poles at $s = 0$ and $s = \infty$. Once proper component functions are chosen, the mathematical procedure for finding the conditions for realizability for each choice of component functions follows. From the previous discussion, the following component functions were selected.

$$z_o(s) = \frac{d_o}{s}$$
$$z_2(s) = d_2\,s \tag{2-25}$$

The remainder function $z_{11}'(s)$ obtained by removing the component functions in (2-25) becomes

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s) \tag{2-26}$$

But, from the two known conditions in (2-23) and (2-24), equation (2-26) yields

$$z_{11}(j\,\omega_\tau) - z_o(j\,\omega_\tau) - z_2(j\,\omega_\tau) = 0 \tag{2-27}$$

$$z_{11}(j\,\omega_{\beta 2}) - z_o(j\,\omega_{\beta 2}) - z_2(j\,\omega_{\beta 2}) = 0 \tag{2-28}$$

Substituting (2-25) into (2-27) and (2-28) gives

$$z_{11}(j\,\omega_\tau) + j\,\frac{d_o}{\omega_\tau} - j\,d_2\,\omega_\tau = 0 \tag{2-29}$$

$$z_{11}(j\,\omega_{\beta 2}) + j\,\frac{d_o}{\omega_{\beta 2}} - j\,d_2\,\omega_{\beta 2} = 0 \qquad (2\text{-}30)$$

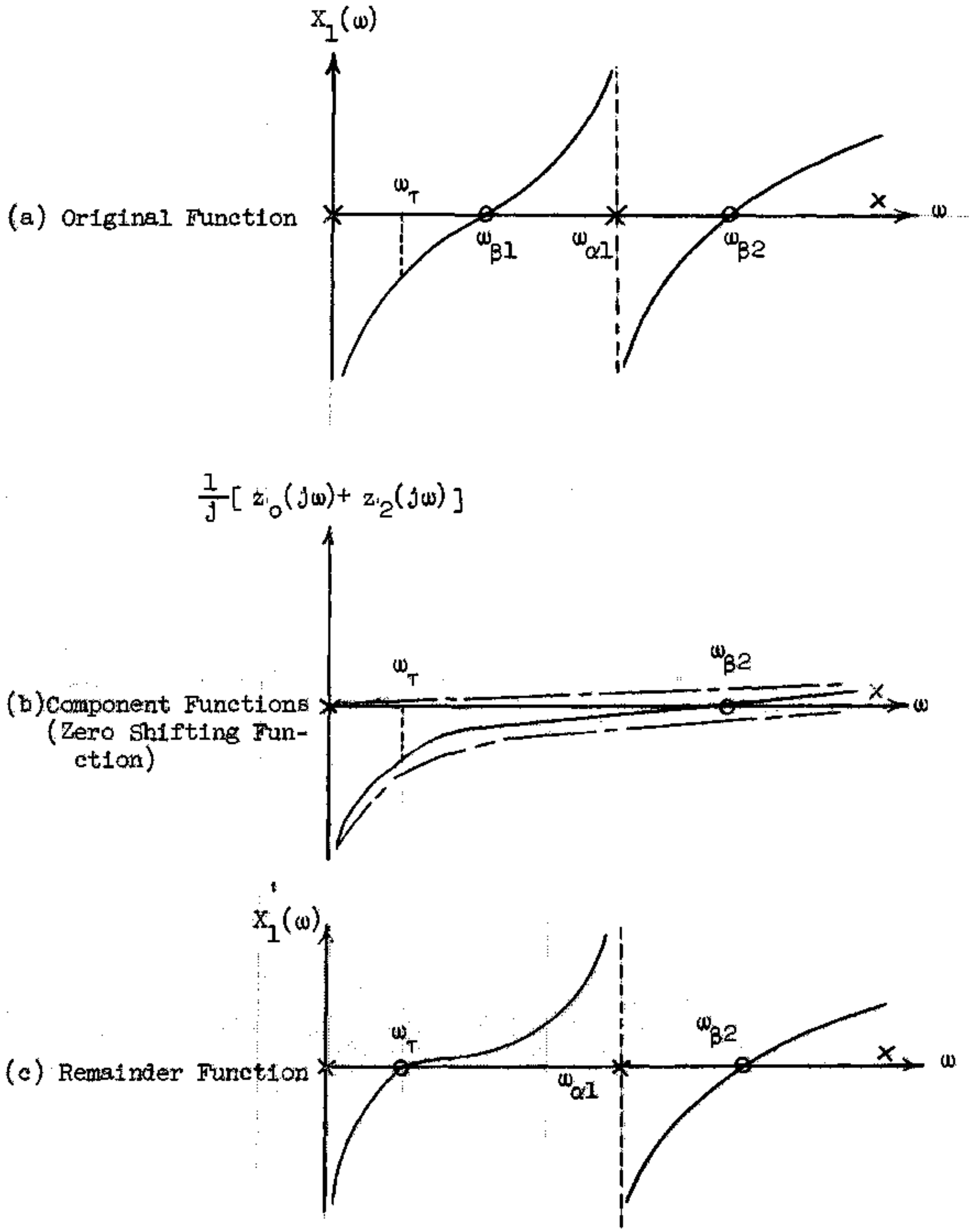

Fig. 14 Graphical Interpretation of the Procedure for Case 1

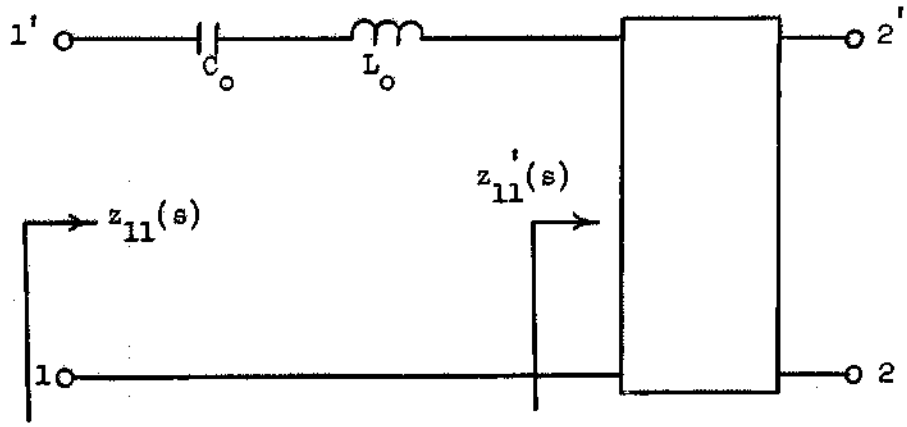

Fig. 15 Network Structure for Case 1

But, since $\omega_{\beta 2}$ is a zero frequency of $z_{11}(s)$,

$$z_{11}(j\,\omega_{\beta 2}) = 0 \tag{2-31}$$

and the equations (2-29) and (2-30) are expressed in matrix form as

$$\begin{bmatrix} -\frac{1}{\omega_\tau} & \omega_\tau \\ -\frac{1}{\omega_{\beta 2}} & \omega_{\beta 2} \end{bmatrix} \cdot \begin{bmatrix} d_o \\ d_2 \end{bmatrix} = \begin{bmatrix} X_1(\omega_\tau) \\ 0 \end{bmatrix} \tag{2-32}$$

Let

$$\triangle = \text{det.} \begin{bmatrix} -\frac{1}{\omega_\tau} & \omega_\tau \\ -\frac{1}{\omega_{\beta 2}} & \omega_{\beta 2} \end{bmatrix}, \quad \triangle_1 = \text{det.} \begin{bmatrix} X_1(j\,\omega_\tau) & \omega_\tau \\ 0 & \omega_{\beta 2} \end{bmatrix},$$

$$\triangle_2 = \text{det.} \begin{bmatrix} -\frac{1}{\omega_\tau} & X_1(j\,\omega_\tau) \\ -\frac{1}{\omega_{\beta 2}} & 0 \end{bmatrix} \tag{2-33}$$

Then, by Cramer's rule, d_o and d_2 yield

$$d_o = \frac{\Delta_1}{\Delta} = \frac{-X_1(\omega_\tau)\ \omega_\tau\ \omega_{\beta 2}^2}{\omega_{\beta 2}^2 - \omega_\tau^2} \tag{2-34}$$

$$d_2 = \frac{\Delta_2}{\Delta} = \frac{-X_1(\omega_\tau)\ \omega_\tau}{\omega_{\beta 2}^2 - \omega_\tau^2} \tag{2-35}$$

The network elements in Fig. 15 are determined uniquely.

$$C_o = \frac{\omega_{\beta 2}^2 - \omega_\tau^2}{-\ X_1(\omega_\tau)\ \omega_\tau\ \omega_{\beta 2}^2} \tag{2-36}$$

$$L_2 = \frac{-\ X_1(\omega_\tau)\ \omega_\tau}{\omega_{\beta 2}^2 - \omega_\tau^2} \tag{2-37}$$

The necessary and sufficient condition for realizability by this procedure is

$$0 < \omega_\tau < \omega_{\beta 1} \tag{2-38}$$

The proof for this condition will be given later.

The graphical interpretation of this procedure is shown in Fig. 14. In that figure, the imaginary impedance level of $z_{11}(s)$ evaluated at $s = j\ \omega_\tau$ and that of the zero shifting function evaluated at $s = j\ \omega_\tau$ are the same. The zero of $z_{11}(s)$ at $s = j\ \omega_{\beta 2}$ is also the zero of zero shifting function. Therefore, by removing the zero shifting section, the remainder function has zeros at $s = j\ \omega_\tau$ and $s = j\ \omega_{\beta 2}$, which completes the procedure.

An alternate method for evaluating d_o and d_2 which is often convenient for solving for d_o and d_2 in (2-29) and (2-30) by computer programming is presented. The remainder function in this case is assumed to have a form as

$$z_{11}'(s) = M \frac{(s^2 + \omega_\tau^2)(s^2 + \omega_{\beta 2}^2)}{S(s^2 + \omega_{\alpha 1}^2)} \tag{2-39}$$

From equation (2-26), $z_{11}'(s)$ was defined by

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s) \tag{2-40}$$

Substituting $z_{11}(s)$, $z_o(s)$ and $z_2(s)$ in equations (2-22) and (2-25) to (2-40) yields

$$M \frac{(s^2 + \omega_\tau^2)(s^2 + \omega_{\beta 2}^2)}{s\ (s^2 + \omega_{\alpha 1}^2)} = K \frac{(s^2 + \omega_{\beta 1}^2)(s^2 + \omega_{\beta 2}^2)}{s\ (s^2 + \omega_{\alpha 1}^2)} - \frac{d_o}{s} - d_2 s \tag{2-41}$$

Multiplying (2-41) by $3(s^2 + \omega_{\alpha 1}^2)$ gives

$$M(s^2 + \omega_\tau^2)(s^2 + \omega_{\beta 2}^2) = K(s^2 + \omega_{\beta 1}^2)(s^2 + \omega_{\beta 2}^2) - d_o(s^2 + \omega_{\alpha 1}^2)$$

$$- d_2\ s^2(s^2 + \omega_{\alpha 1}^2) \tag{2-42}$$

In equation (2-42), all the parameters except M, d_o and d_2 are known. To solve for the unknown parameters in (2-42), let $s^2 \to \omega_{\alpha 1}^2$, then

$$M(\omega_\tau^2 - \omega_{\alpha 1}^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2) = K(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)$$

or

$$M = \frac{K(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)}{(\omega_T^2 - \omega_{\alpha 1}^2)} \tag{2-43}$$

Let $s^2 \to 0$ in (2-42), then

$$M\,\omega_T^2\,\omega_{\beta 2}^2 = K\,\omega_{\beta 1}^2\,\omega_{\beta 2}^2 - d_o\,\omega_{\alpha 1}^2$$

or

$$d_o = \frac{M\,\omega_T^2\,\omega_{\beta 2}^2 - K\,\omega_{\beta 1}^2\,\omega_{\beta 2}^2}{-\,\omega_{\alpha 1}^2} \tag{2-44}$$

Again, let $s^2 \to -\,\omega_T^2$ in (2-42), then

$$0 = K(\omega_{\beta 1}^2 - \omega_T^2)(\omega_{\beta 2}^2 - \omega_T^2) + d_o(\omega_{\alpha 1}^2 - \omega_T^2) + d_2(-\omega_T^2)(\omega_{\alpha 1}^2 - \omega_T^2)$$

or

$$d_2 = \frac{K(\omega_{\beta 1}^2 - \omega_T^2)(\omega_{\beta 2}^2 - \omega_T^2) + d_o(\omega_{\alpha 1}^2 - \omega_T^2)}{-\,\omega_T^2\,(\omega_{\alpha 1}^2 - \omega_T^2)} \tag{2-45}$$

From (2-43), (2-44) and (2-45), M, d_o and d_2 are

$$M = \frac{K(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)}{(\omega_T^2 - \omega_{\alpha 1}^2)} \tag{2-46}$$

$$d_o = K\,\frac{\omega_{\beta 2}^2(\omega_{\beta 1}^2 - \omega_T^2)}{(\omega_{\alpha 1}^2 - \omega_T^2)} \tag{2-47}$$

$$d_2 = K\,\frac{(\omega_{\beta 1}^2 - \omega_T^2)}{(\omega_{\alpha 1}^2 - \omega_T^2)} \tag{2-48}$$

which determines the network elements C_o and L_2 in Fig. 15.

This method can be applied to any 2-z function and also to higher order functions. The results in (2-47) and (2-48) are essentially the same as the previous ones if $X_1(\omega_T) = \dfrac{-(\omega_{\beta 1}^2 - \omega_T^2)(\omega_{\beta 2}^2 - \omega_T^2)}{\omega_T(\omega_{\alpha 1}^2 - \omega_T^2)}$ is substituted in (2-36) and (2-37).

The choice of control frequencies in this procedure is unique because all the combinations of control frequencies other than the one of the poles at s=0 and s=oo become unrealizable. For example, suppose that the poles at s=0 and $s=j\,\omega_{\alpha 1}$ are chosen for control frequencies. Then, the component functions chosen in this case are

$$z_o(s) = \frac{d_o}{s} \tag{2-49}$$

$$z_1(s) = \frac{d_1\, s}{s^2 + \omega_{\alpha 1}^2} \tag{2-50}$$

Since the remainder function $z_{11}'(s)$ has zeros at $s=j\,\omega_T$ and $s=j\,\omega_{\beta 2}$, the following two relationships must be satisfied.

$$z_{11}(s) - z_o(s) - z_1(s)\Big|_{s=j\,\omega_T} = 0 \tag{2-51}$$

$$z_{11}(s) - z_o(s) - z_1(s)\Big|_{s=j\,\omega_{\beta 2}} = 0 \tag{2-52}$$

Substituting (2-49) and (2-50) in (2-51) and (2-52) leads to the matrix equation

$$\begin{bmatrix} -\dfrac{1}{\omega_\tau} & \dfrac{\omega_\tau}{\omega_{\alpha 1}^2 - \omega_\tau^2} \\ \\ -\dfrac{1}{\omega_{\beta 2}} & \dfrac{\omega_{\beta 2}}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} \end{bmatrix} \cdot \begin{bmatrix} d_o \\ \\ d_1 \end{bmatrix} = \begin{bmatrix} X_1(\omega_\tau) \\ \\ 0 \end{bmatrix} \qquad (2\text{-}53)$$

Solving for d_o and d_1 yields

$$d_o = \frac{-\omega_\tau\ \omega_{\beta 2}^2(\omega_{\alpha 1}^2 - \omega_\tau^2)\ X_1(\omega_\tau)}{\omega_{\alpha 1}^2(\omega_{\beta 2}^2 - \omega_\tau^2)} \qquad (2\text{-}54)$$

$$d_1 = \frac{\omega_\tau(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_{\alpha 1}^2 - \omega_\tau^2)\ X_1(\omega_\tau)}{\omega_{\alpha 1}^2(\omega_{\beta 2}^2 - \omega_\tau^2)} \qquad (2\text{-}55)$$

But

$$X_1(\omega_\tau) < 0 \qquad (2\text{-}56)$$

and

$$0 < \omega_\tau < \omega_{\beta 1}$$

Therefore, d_1 in (2-55) has a negative value and this procedure is not realizable.

The other case, that is to choose the poles at $s = j\ \omega_{\alpha 1}$ and $s = \infty$ for control frequencies, also becomes unrealizable because the value of d_2 in the Type III-like component function becomes negative.

Case 2; $\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1}$

By analogy with Case 1, the two known conditions are

$$z_{11}'(j\ \omega_\tau) = 0 \qquad (2\text{-}49)$$

$$z_{11}'(j\,\omega_{\beta 2}) = 0 \tag{2-50}$$

The conditions described above are imposed because the remainder function $z_{11}'(s)$ has zeros at $s=j\,\omega_T$ and $s=j\,\omega_{\beta 2}$ after removing zero shifting sections. To choose the proper component functions (or control frequencies) for the zero shifting section, the effect of removing each possible component function must be investigated. There are three possible control frequencies; poles at $s=0$, $s=j\,\omega_{\alpha 1}$, and $s=\infty$. One of these three control frequencies may not be taken for a control frequency, since only two known conditions are available.

Of the three zero shifting techniques discussed in Art. 2.1, removal of Type I and III sections shifts the zero at $s=j\,\omega_{\beta 1}$ away from the origin. Again, Type I and III sections are identified as zero shifting sections which have control frequencies at $s=\infty$ and $s=j\,\omega_{\alpha 1}$. The partial removal of the residue in the finite pole at $s=j\,\omega_{\alpha 1}$ results in shifting the zero at $s=j\,\omega_{\beta 2}$ towards the origin. This can be compensated for by removal of Type I sections to retain the zero at $s=j\,\omega_{\beta 2}$. If a Type II section which has a control frequency at $s=0$ is removed for the proposed objective, two additional component functions are required and they cannot be determined. Thus, two control frequencies at $s=\infty$ and $s=j\,\omega_{\alpha 1}$ are chosen for Case 2 and their component functions are

$$z_1(s) = \frac{d_1 s}{s^2 + \omega_{\alpha 1}^2} \tag{2-51}$$

$$z_2(s) = d_2\, s \tag{2-52}$$

Now, the problem is to determine proper values of d_1 and d_2 in (2-51) and (2-52) such that the remainder function has the desired zeros. From the two known conditions in (2-49) and (2-50), two equations which must be satisfied are

$$z_{11}(s) - z_1(s) - z_2(s) \Big|_{s=j\,\omega_\tau} = 0 \tag{2-53}$$

$$z_{11}(s) - z_1(s) - z_2(s) \Big|_{s=j\,\omega_{\beta 2}} = 0 \tag{2-54}$$

Substituting (2-51) and (2-52) in (2-53) and (2-54) gives

$$z_{11}(j\,\omega_\tau) - \frac{jd_1\omega_\tau}{\omega_{\alpha 1}^2 - \omega_\tau^2} - jd_2\,\omega_\tau = 0 \tag{2-55}$$

$$z_{11}(j\,\omega_{\beta 2}) - \frac{jd_1\,\omega_{\beta 2}}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} - j\,d_2\,\omega_{\beta 2} = 0 \tag{2-56}$$

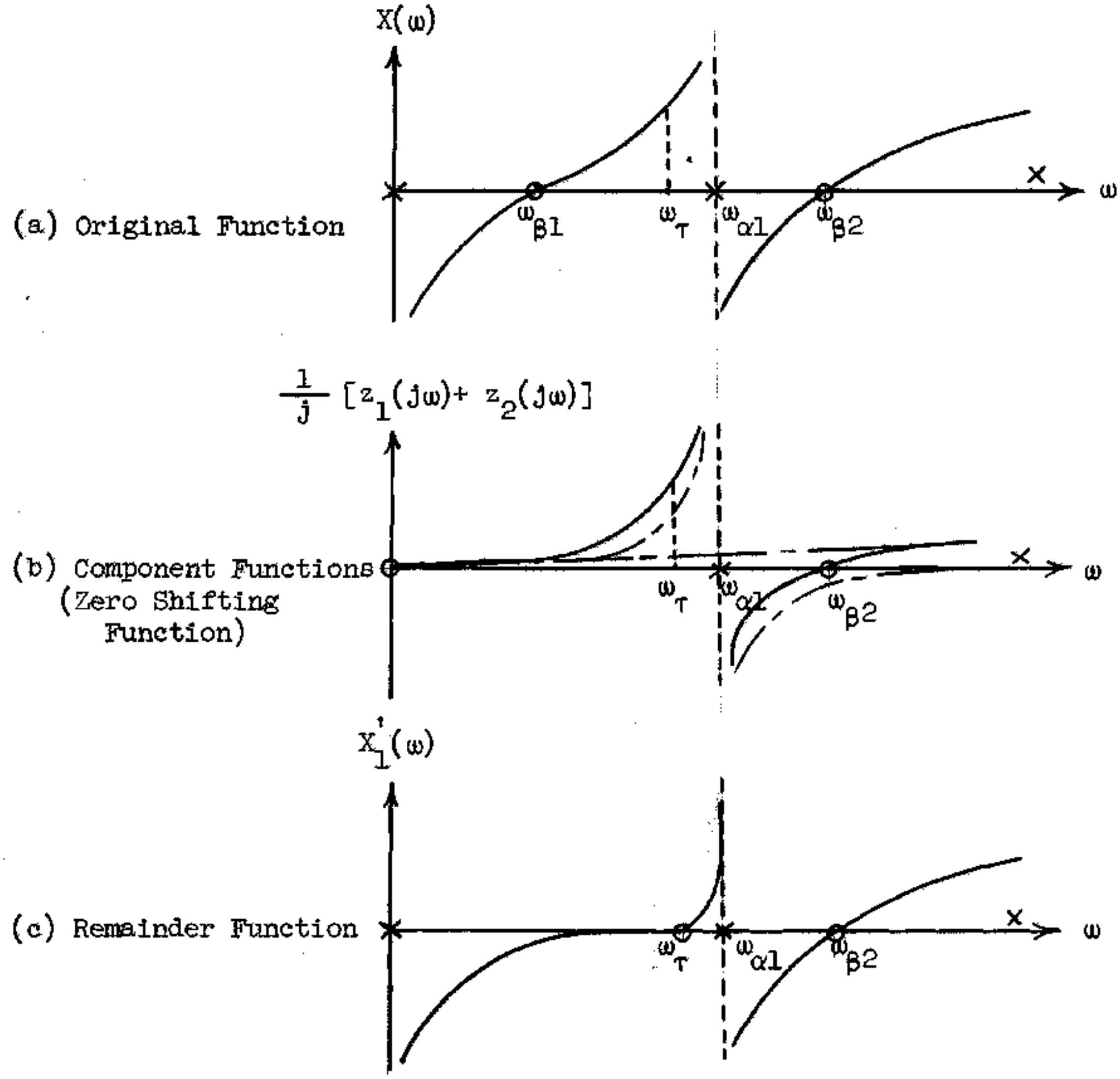

Fig. 16 Graphical Interpretation of the Procedure for Case 2

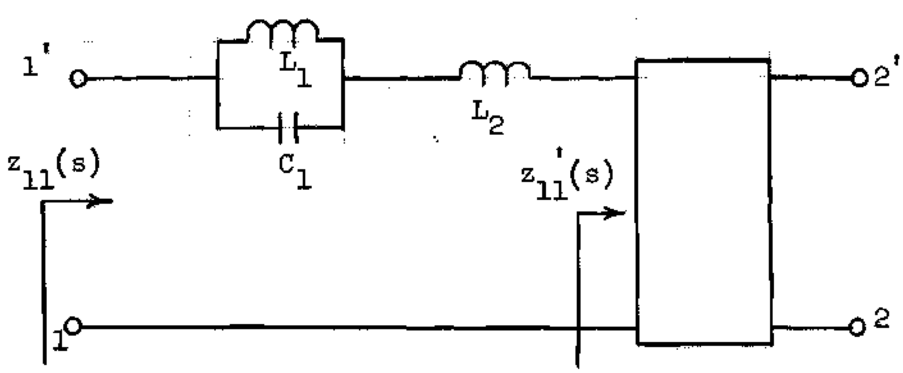

Fig. 17 Network Structure for Case 2

But, since $z_{11}(s)$ is a pure reactance function and has a zero at $s = j\,\omega_{\beta 2}$,

$$z_{11}(j\,\omega_\tau) = j\,X_1(\omega_\tau), \quad z_{11}(j\,\omega_{\beta 2}) = 0 \tag{2-57}$$

The matrix form for (2-55) and (2-56) is

$$\begin{bmatrix} \dfrac{\omega_\tau}{\omega_{\alpha 1}^2 - \omega_\tau^2} & \omega_\tau \\ \\ \dfrac{\omega_{\beta 2}}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} & \omega_{\beta 2} \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ \\ d_2 \end{bmatrix} = \begin{bmatrix} X_1(\omega_\tau) \\ \\ 0 \end{bmatrix} \tag{2-58}$$

Let (as in Case 1)

$$\Delta = \det. \begin{bmatrix} \dfrac{\omega_\tau}{\omega_{\alpha 1}^2 - \omega_\tau^2} & \omega_\tau \\ \\ \dfrac{\omega_{\beta 2}}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} & \omega_{\beta 2} \end{bmatrix}, \quad \Delta_1 = \det. \begin{bmatrix} X_1(\omega_\tau) & \omega_\tau \\ \\ 0 & \omega_{\beta 2} \end{bmatrix},$$

$$\Delta_2 = \det. \begin{bmatrix} \dfrac{\omega_\tau}{\omega_{\alpha 1}^2 - \omega_\tau^2} & X_1(\omega_\tau) \\ \\ \dfrac{\omega_{\beta 2}}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} & 0 \end{bmatrix} \tag{2-59}$$

By Cramer's rule, d_1 and d_2 become

$$d_1 = \frac{\Delta_1}{\Delta} = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)}{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2)} X_1(\omega_\tau) \tag{2-60}$$

$$d_2 = \frac{\Delta_2}{\Delta} = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)}{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2} X_1(\omega_\tau) \tag{2-61}$$

The network elements in Fig. 17 are determined as

$$C_1 = \frac{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2)}{(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2) X_1(\omega_\tau)} \tag{2-62}$$

$$L_1 = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)}{\omega_\tau\, \omega_{\alpha 1}^2(\omega_{\beta 2}^2 - \omega_\tau^2)} X_1(\omega_\tau) \tag{2-63}$$

$$L_2 = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)}{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2)} X_1(\omega_\tau) \tag{2-64}$$

The remainder function $z_{11}'(s)$ can be obtained by subtracting $z_1(s)$ and $z_2(s)$ with known d_1 and d_2 from $z_{11}(s)$, and this $z_{11}'(s)$ has zeros at $s=j\,\omega_\tau$ and $s=j\,\omega_{\beta 2}$. The procedure for Case 2 is completed. The necessary and sufficient condition for realizability by this procedure is

$$\omega_{\beta 1} > \omega_\tau > \omega_{\alpha 1} \tag{2-65}$$

The proof for this condition will be given later.

The graphical interpretation of this procedure is shown in Fig. 16.

In that figure, the zero shifting function has the same impedance level as $z_{11}(s)$ at $s=j\ \omega_T$ and has a zero at $s=j\ \omega_{\beta 2}$. The zero shifting function is composed of two component functions $z_1(s)$ and $z_2(s)$. By removing the zero shifting section, the remainder function $z_{11}'(s)$ has zeros at $s=j\ \omega_T$ and $s=j\ \omega_{\beta 2}$, which completes the procedure.

An alternate method of determining d_1 and d_2 can be applied to this case with a scheme similar to the one discussed in Case 1.

Also, the uniqueness of choice of control frequencies is guaranteed since the other choices are not realizable. For example, if poles at s=0 and s=oo are chosen for Case 2, from (2-34) and (2-35), $X_1(\omega_T) < 0$ is required to make both d_o and d_2 positive. But for Case 2, $X_1(\omega_T) > 0$ and the procedure with control frequencies at s=0 and s=oo is not realizable.

Case 3; $\omega_{\alpha 1} < \omega_T < \omega_{\beta 2}$

The proposed problem in this case is to shift a zero at $s=j\ \omega_{\beta 2}$ to $s=j\ \omega_T$ and to retain the other zero location at $s=j\ \omega_{\beta 1}$ the same. Thus, two relationships in (2-23) and (2-24) are changed to

$$z_{11}'(j\ \omega_{\beta 1}) = 0 \tag{2-66}$$

$$z_{11}'(j\ \omega_T) = 0 \tag{2-67}$$

where $z_{11}'(s)$ is a remainder function.

To choose proper control frequencies, each zero shifting technique in Art. 2.1 must be examined. A Type III section, which was defined by the partial removal of the residue in the pole at $s=j\ \omega_{\alpha 1}$, gives not only the shiftings of a zero at $s=j\ \omega_{\beta 2}$ toward the origin but also the shifting of a zero at $s=j\,\omega_{\beta 1}$ away from the origin. In order to retain the zero at $s=j\,\omega_{\beta 1}$, another component function for a zero shifting must be used to compensate the displacement of the zero at $s=j\,\omega_{\beta 1}$ which is occurred by removing a Type III section. Since the partial removal of residue in the pole at s=0 shifts all the zeros toward the origin, a Type II section can be used to readjust the locations of new zeros. However, since removing Type I sections gives zero shiftings away from the origin, a pole at infinity may not be taken for a control frequency. For these reasons, poles at s=0 and $s=j\,\omega_{\alpha 1}$ are chosen for control frequencies. Then, component functions with chosen control frequencies are

$$z_o(s) = \frac{d_o}{s} \tag{2-68}$$

$$z_1(s) = \frac{d_1 s}{s^2 + \omega_{\alpha 1}^2} \tag{2-69}$$

By a similar procedure as in Case 1 or Case 2, a matrix equation to determine d_o and d_1 in (2-68) and (2-69) for this case becomes

$$\begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega_{\alpha 1}^2 - \omega_{\beta 1}^2} \\ -\dfrac{1}{\omega_T} & \dfrac{\omega_T}{\omega_{\alpha 1}^2 - \omega_T^2} \end{bmatrix} \cdot \begin{bmatrix} d_o \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ X_1(\omega_T) \end{bmatrix} \tag{2-70}$$

From (2-70), d_o and d_1 are determined uniquely and the procedure is completed. The condition to be realizable for this case can be found in Table 1.

Case 4; $\omega_{\beta 2} < \omega_T < \infty$

The same procedure given in Case 3 can be applied to Case 4 with the exception of choosing control frequencies. By similar reasoning to that of Case 1, poles at $s=0$ and $s=\infty$ can be chosen for control frequencies. The realizability condition for this case can be found in Table 1.

Table 1 Control Frequencies and Conditions for Realizability in a 2-Z Function

Cases	Control frequencies	Conditions for realizability
1	Poles at $s=0$ $s=\infty$	$0 < \omega_T < \omega_{\beta 1}$
2	Poles at $s=j\,\omega_{\alpha 1}$ $s=\infty$	$\omega_{\beta 1} < \omega_T < \omega_{\alpha 1}$
3	Poles at $s=0$ $s=j\,\omega_{\alpha 1}$	$\omega_{\alpha 1} < \omega_T < \omega_{\beta 2}$
4	Poles at $s=0$ $s=\infty$	$\omega_{\beta 2} < \omega_T < \infty$

As shown in Table 1, shifting any zero of a 2-z reactance function and leaving the other zero the same is always possible provided the frequency of the zero to be shifted is not the same as that of the pole of $z_{11}(s)$ itself. This can be achieved only through proper choice of control frequencies. Therefore, the success (or failure) of this procedure depends upon the proper choice of control frequencies.

Procedure for 3-z Functions

For a 3-z function, the impedance function $z_{11}(s)$ has the form

$$z_{11}(s) = K \frac{(s^2+\omega_{\beta 1}^2)(s^2+\omega_{\beta 2}^2)(s^2+\omega_{\beta 3}^2)}{s\,(s^2+\omega_{\alpha 1}^2)(s^2+\omega_{\alpha 2}^2)} \tag{2-71}$$

There are four poles that are available for control frequencies; poles at $s=0$, $s=j\,\omega_{\alpha 1}$, $s=j\,\omega_{\alpha 2}$ and $s=\infty$. There are also six positions at which ω_τ can lie on the positive real frequency axis; $0 < \omega_\tau < \omega_{\beta 1}$, $\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1}$, $\omega_{\alpha 1} < \omega_\tau < \omega_{\beta 2}$, $\omega_{\beta 2} < \omega_\tau < \omega_{\alpha 2}$, $\omega_{\alpha 2} < \omega_\tau < \omega_{\beta 3}$ and $\omega_{\beta 3} < \omega_\tau < \infty$.

Because of the similarity between procedures for each situation, only the case of $\omega_{\beta 2} < \omega_\tau < \omega_{\alpha 2}$ will be discussed; the other cases can be found in Table 2.

Now, consider the case of $\omega_{\beta 2} < \omega_\tau < \omega_{\alpha 2}$. The proposed problem in this case is to shift a zero at $s=j\,\omega_{\beta 2}$ to $s=j\,\omega_\tau$ and leave all the other zeros the same. Thus, after removing zero shifting sections, the remainder function $z_{11}'(s)$ must have zeros at $s=j\,\omega_{\beta 1}$, $s=j\,\omega_\tau$ and $s=j\,\omega_{\beta 3}$, which can be expressed analytically as

$$z_{11}'(j\,\omega_{\beta 1}) = 0\,, \quad z_{11}'(j\,\omega_\tau) = 0 \quad \text{and}$$
$$z_{11}'(j\,\omega_{\beta 3}) = 0 \tag{2-72}$$

Since three known conditions are available as in (2-72), at most three component functions (or control frequencies) must be taken for this procedure, otherwise they are not determined. There are altogether four possible combinations of control frequencies to be encountered; poles at $s=0$, $s=j\,\omega_{\alpha 1}$ and $s=j\,\omega_{\alpha 2}$; poles at $s=j\,\omega_{\alpha 1}$, $s=j\,\omega_{\alpha 2}$ and $s=\infty$; poles at $s=0$, $s=j\,\omega_{\alpha 1}$ and $s=\infty$; poles at $s=0$, $s=j\,\omega_{\beta 2}$ and $s=\infty$. Each combination will be examined separately.

(a) If poles at $s=0$, $s=j\,\omega_{\alpha 1}$ and $s=j\,\omega_{\alpha 2}$ are chosen for control frequencies, a zero at $s=j\,\omega_{\beta 3}$ will be shifted toward the origin and an additional control frequency is required to shift the zero back to original location. This additional component function cannot be specified, since only three conditions are available.

(b) If poles at $s=j\,\omega_{\alpha 1}$, $s=j\,\omega_{\alpha 2}$ and $s=\infty$ are chosen for control frequencies, a zero at $s=j\,\omega_{\beta 1}$ will be shifted away from the origin and also an additional control frequency is required, whose component function cannot be specified.

(c) If poles at $s=0$, $s=j\,\omega_{\alpha 1}$ and $s=\infty$ are chosen for control frequencies, a zero at $s=j\,\omega_{\beta 2}$ tends to be shifted toward the origin rather than away from the origin. Thus, component functions with those control frequencies are often unrealizable.

(d) If poles at $s=0$, $s=j\,\omega_{\alpha 2}$ and $s=\infty$ are taken for control frequencies, the proposed zero shiftings can be achieved if the partial residues to be removed are properly selected.

From the above four observations, poles at $s=0$, $s=j\,\omega_{\alpha 2}$ and $s=\infty$ are chosen for control frequencies and their component functions are

$$z_o(s) = \frac{d_o}{s} \qquad (2\text{-}73)$$

$$z_2(s) = \frac{d_2 s}{s^2 + \omega_{\alpha 2}^2} \tag{2-74}$$

and

$$z_3(s) = d_3\ s \tag{2-75}$$

Thus, the remainder function $z_{11}'(s)$ is defined as

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s) - z_3(s) \tag{2-76}$$

From (2-72) and (2-76), three equations which must be satisfied are

$$z_{11}(j\ \omega_{\beta 1}) - z_o(j\ \omega_{\beta 1}) - z_2(j\ \omega_{\beta 1}) - z_3(j\ \omega_{\beta 1}) = 0 \tag{2-77}$$

$$z_{11}(j\ \omega_{\tau}) - z_o(j\ \omega_{\tau}) - z_2(j\ \omega_{\tau}) - z_3(j\ \omega_{\tau}) = 0 \tag{2-78}$$

$$z_{11}(j\ \omega_{\beta 3}) - z_o(j\ \omega_{\beta 3}) - z_2(j\ \omega_{\beta 3}) - z_3(j\ \omega_{\beta 3}) = 0 \tag{2-79}$$

Substituting (2-73), (2-74) and (2-75) into (2-77), (2-78) and (2-79) gives

$$\begin{aligned} +j\ \frac{d_o}{\omega_{\beta 1}} - j\ \frac{d_2\ \omega_{\beta 1}}{\omega_{\alpha 2}^2 - \omega_{\beta 1}^2} - j\ d_3\ \omega_{\beta 1} &= 0 \\ z_{11}(j\ \omega_\tau) + j\ \frac{d_o}{\omega_\tau} - j\ \frac{d_2\ \omega_\tau}{\omega_{\alpha 2}^2 - \omega_\tau^2} - j\ d_3\ \omega_\tau &= 0 \\ +j\ \frac{d_o}{\omega_{\beta 3}} - j\ \frac{d_3\ \omega_{\beta 3}}{\omega_{\alpha 2}^2 - \omega_{\beta 3}^2} - j\ d_3\ \omega_{\beta 3} &= 0 \end{aligned} \tag{2-80}$$

Since

$z_{11}(j\ \omega_\tau) = j\ X_1(\omega_\tau)$, the three simultaneous equations in (2-80) can be expressed in a matrix form as

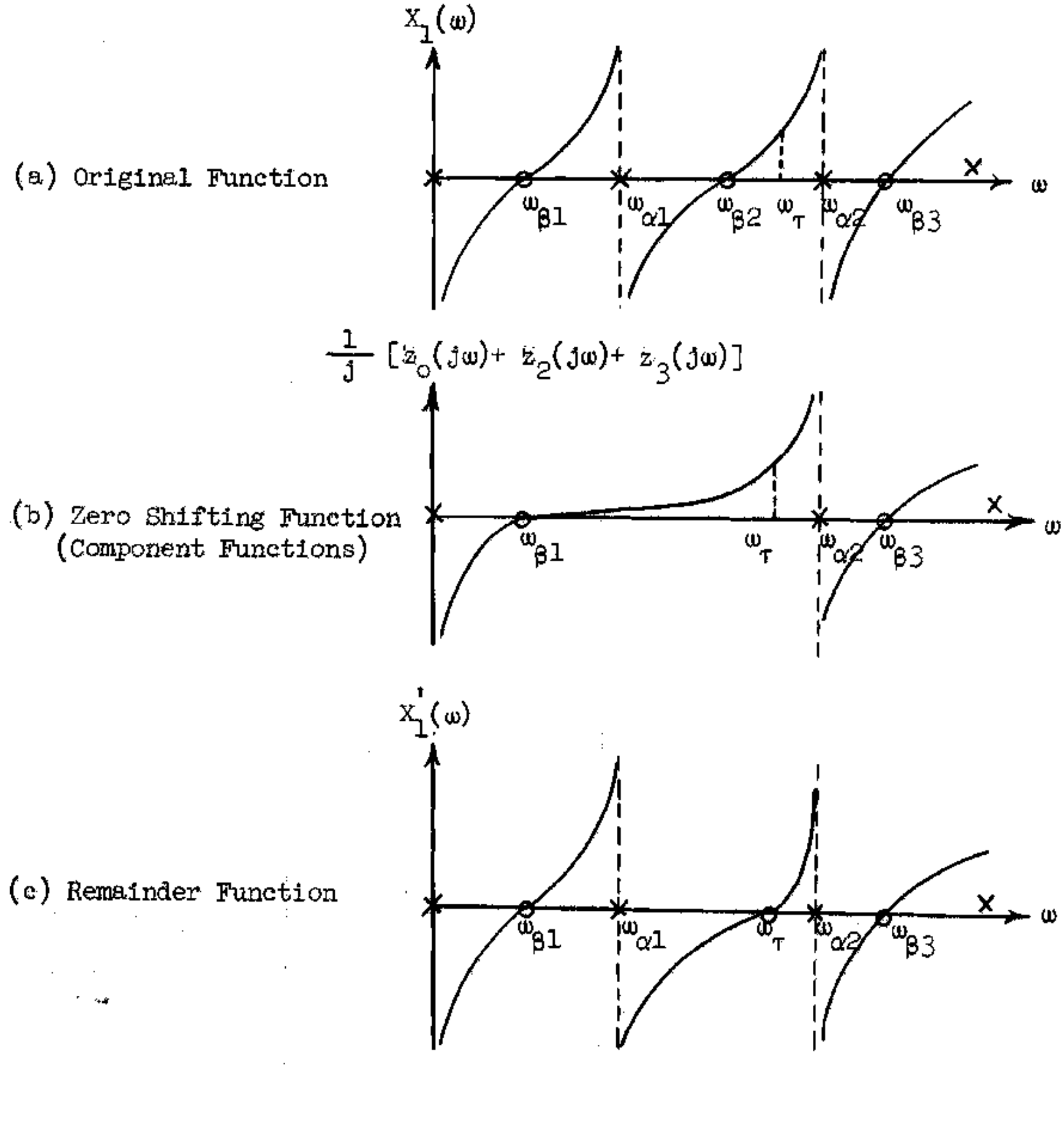

Fig. 18 Graphical Interpretation of the Procedure for a 3-z Function

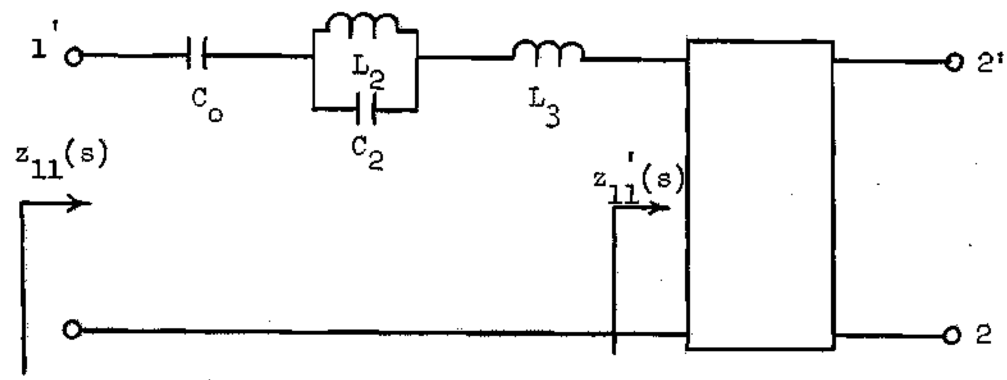

Fig. 19 Network Structure for a 3-z Function

$$\begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega_{\alpha 2}^2 - \omega_{\beta 1}^2} & \omega_{\beta 1} \\ -\dfrac{1}{\omega_{\tau}} & \dfrac{\omega_{\tau}}{\omega_{\alpha 2}^2 - \omega_{\tau}^2} & \omega_{\tau} \\ -\dfrac{1}{\omega_{\beta 3}} & \dfrac{\omega_{\beta 3}}{\omega_{\alpha 2}^2 - \omega_{\beta 3}^2} & \omega_{\beta 3} \end{bmatrix} \cdot \begin{bmatrix} d_o \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ X_1(\omega_\tau) \\ 0 \end{bmatrix} \qquad (2\text{-}81)$$

Let Δ, Δ_1, Δ_2 and Δ_3 be denoted as

$$\Delta = \det. \begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega_{\alpha 2}^2 - \omega_{\beta 1}^2} & \omega_{\beta 1} \\ -\dfrac{1}{\omega_{\tau}} & \dfrac{\omega_{\tau}}{\omega_{\alpha 2}^2 - \omega_{\tau}^2} & \omega_{\tau} \\ -\dfrac{1}{\omega_{\beta 3}} & \dfrac{\omega_{\beta 3}}{\omega_{\alpha 2}^2 - \omega_{\beta 3}^2} & \omega_{\beta 3} \end{bmatrix}, \qquad (2\text{-}82)$$

$$\triangle_1 = \det. \begin{bmatrix} 0 & \dfrac{\omega_{\beta 1}}{\omega_{\alpha 2}^2 - \omega_{\beta 1}^2} & \omega_{\beta 1} \\ X_1(\omega_\tau) & \dfrac{\omega_\tau}{\omega_{\alpha 2}^2 - \omega_\tau^2} & \omega_\tau \\ 0 & \dfrac{\omega_{\beta 3}}{\omega_{\alpha 2}^2 - \omega_{\beta 3}^2} & \omega_{\beta 3} \end{bmatrix}, \tag{2-82}$$

$$\triangle_2 = \det. \begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & 0 & \omega_{\beta 1} \\ -\dfrac{1}{\omega_\tau} & X_1(\omega_\tau) & \omega_\tau \\ -\dfrac{1}{\omega_{\beta 3}} & 0 & \omega_{\beta 3} \end{bmatrix}, \quad \triangle_3 = \det. \begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega_{\alpha 2}^2 - \omega_{\beta 1}^2} & 0 \\ -\dfrac{1}{\omega_\tau} & \dfrac{\omega_\tau}{\omega_{\alpha 2}^2 - \omega_\tau^2} & X_1(\omega_\tau) \\ -\dfrac{1}{\omega_{\beta 3}} & \dfrac{\omega_{\beta 3}}{\omega_{\alpha 2}^2 - \omega_{\beta 3}^2} & 0 \end{bmatrix} \tag{2-82}$$

Then, d_o, d_2 and d_3 are determined by Cramer's rule.

$$d_o = \frac{\triangle_1}{\triangle}, \qquad d_2 = \frac{\triangle_2}{\triangle}, \qquad d_3 = \frac{\triangle_3}{\triangle} \tag{2-83}$$

The d's in (2-83) define the component functions and the remainder function $z_{11}'(s)$ can be obtained as

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s) - z_3(s) \tag{2-84}$$

which has zeros at $s=j\,\omega_{\beta 1}$, $s=j\,\omega_\tau$, and $s=j\,\omega_{\beta 3}$.

An alternate method to evaluate d_o, d_2 and d_3 is presented. By analogy with Case 1 in the 2-z function, the remainder function $z_{11}'(s)$ is assumed to be

$$z_{11}'(s) = M\,\frac{(s^2+\omega_{\beta1}^2)(s^2+\omega_{\tau}^2)(s^2+\omega_{\beta3}^2)}{s(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2)} \tag{2-85}$$

By (2-84), $Z_{11}'(s)$ was defined as

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s) - z_3(s) \tag{2-86}$$

Substitution of $z_{11}(s)$ in (2-71), $z_o(s)$ in (2-73), $z_2(s)$ in (2-74), and $z_3(s)$ in (2-75) into (2-86) gives

$$M\,\frac{(s^2+\omega_{\beta1}^2)(s^2+\omega_{\tau}^2)(s^2+\omega_{\beta3}^2)}{s\,(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2)} = K\,\frac{(s^2+\omega_{\beta1}^2)(s^2+\omega_{\beta2}^2)(s^2+\omega_{\beta3}^2)}{s\,(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2)} - \frac{d_o}{s} - \frac{d_2 s}{s^2+\omega_{\alpha2}^2} - d_3 s \tag{2-87}$$

Multiplying (2-87) by $s(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2)$ yields

$$\begin{aligned} M(s^2+\omega_{\beta1}^2)(s^2+\omega_{\tau}^2)(s^2+\omega_{\beta3}^2) &= K(s^2+\omega_{\beta1}^2)(s^2+\omega_{\beta2}^2)(s^2+\omega_{\beta3}^2) \\ &\quad - d_o(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2) - d_2 s^2(s^2+\omega_{\alpha1}^2) \\ &\quad - d_3 s^2(s^2+\omega_{\alpha1}^2)(s^2+\omega_{\alpha2}^2) \ldots\ldots\ldots \end{aligned} \tag{2-88}$$

In equation (2-88), all the parameters except M, d_o, d_2 and d_3 are known. To solve for the unknown parameters in (2-88) step-by-step, Let $s^2 \to -\omega_{\alpha 1}^2$, then

$$M(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)(\omega_\tau^2 - \omega_{\alpha 1}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 1}^2) = K(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 1}^2)$$

or

$$M = K\frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)}{(\omega_\tau^2 - \omega_{\alpha 1}^2)} \tag{2-89}$$

Let $s^2 \to 0$ in (2-88), then

$$M\,\omega_{\beta 1}^2\,\omega_\tau^2\,\omega_{\beta 3}^2 = K\,\omega_{\beta 1}^2\,\omega_{\beta 2}^2\,\omega_{\beta 3}^2 - d_o\,\omega_{\alpha 1}^2\,\omega_{\alpha 2}^2$$

or

$$d_o = \frac{K\,\omega_{\beta 1}^2\,\omega_{\beta 2}^2\,\omega_{\beta 3}^2 - M\,\omega_{\beta 1}^2\,\omega_\tau^2\,\omega_{\beta 3}^3}{\omega_{\alpha 1}^2\,\omega_{\alpha 2}^2} \tag{2-90}$$

Let $s^2 \to -\omega_{\alpha 2}^2$ in (2-88), then

$$M(\omega_{\beta 1}^2 - \omega_{\alpha 2}^2)(\omega_\tau^2 - \omega_{\alpha 2}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 2}^2) = K(\omega_{\beta 1}^2 - \omega_{\alpha 2}^2)(\omega_{\beta 2}^2 - \omega_{\alpha 2}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 2}^2)$$
$$- d_2(-\omega_{\alpha 2}^2)(\omega_{\alpha 1}^2 - \omega_{\alpha 2}^2)$$

or

$$d_2 = \frac{(\omega_{\beta 1}^2 - \omega_{\alpha 2}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 2}^2)\left[M(\omega_\tau^2 - \omega_{\alpha 2}^2) - K(\omega_{\beta 2}^2 - \omega_{\alpha 2}^2)\right]}{\omega_{\alpha 2}^2(\omega_{\alpha 1}^2 - \omega_{\alpha 2}^2)} \tag{2-91}$$

Again, let $s^2 \to -\omega_\tau^2$ in (2-88), then

$$0 = K(\omega_{\beta 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_\tau^2)(\omega_{\beta 3}^2 - \omega_\tau^2) - d_o(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\alpha 2}^2 - \omega_\tau^2)$$
$$- d_2(-\omega_\tau^2)(\omega_{\alpha 1}^2 - \omega_\tau^2) - d_3(-\omega_\tau^2)(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\alpha 2}^2 - \omega_\tau^2)$$

or

$$d_3 = \frac{d_o(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\alpha 2}^2 - \omega_\tau^2) - d_2\,\omega_\tau^2(\omega_{\alpha 1}^2 - \omega_\tau^2) - K(\omega_{\beta 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_\tau^2)(\omega_{\beta 3}^2 - \omega_\tau^2)}{\omega_\tau^2(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\alpha 2}^2 - \omega_\tau^2)}$$

.... (2-92)

From (2-89), (2-90), (2-91) and (2-92), M, d_o, d_2 and d_3 are

$$M = K\,\frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)}{(\omega_\tau^2 - \omega_{\alpha 1}^2)} \tag{2-93}$$

$$d_o = K\,\frac{\omega_{\beta 1}^2\,\omega_{\beta 3}^2(\omega_\tau^2 - \omega_{\beta 2}^2)}{\omega_{\alpha 2}^2(\omega_\tau^2 - \omega_{\alpha 1}^2)} \tag{2-94}$$

$$d_2 = K\,\frac{(\omega_{\beta 1}^2 - \omega_{\alpha 2}^2)(\omega_{\beta 3}^2 - \omega_{\alpha 2}^2)(\omega_{\beta 2}^2 - \omega_\tau^2)}{\omega_{\alpha 2}^2(\omega_\tau^2 - \omega_{\alpha 1}^2)} \tag{2-95}$$

$$d_3 = K\,\frac{(\omega_\tau^2 - \omega_{\beta 2}^2)}{(\omega_\tau^2 - \omega_{\alpha 1}^2)} \tag{2-96}$$

which determine the network elements in Fig. 19.

A necessary condition for realizability by this procedure is

$$\omega_{\beta 2} < \omega_\tau < \omega_{\alpha 2} \tag{2-97}$$

The justification of statement in (2-97) follows:

By contradition, if any of the inequalities in (2-97) is not satisfied, from (2-95) and (2-96),

$$d_2 < 0 \quad \text{and } d_3 < 0. \tag{2-98}$$

Since only passive elements are allowed in this process, component functions with negative d_2 and d_3 are not realizable. Therefore, in order for this procedure to be realizable, the condition in (2-97) must be satisfied.

A graphical interpretation of this process is shown in Fig. 18. In that figure, the zero shifting function, which is composed of three component functions, has zeros at $s=j\,\omega_{\beta 1}$ and $s=j\,\omega_{\beta 3}$ and also has the same impedance level at $s=j\,\omega_{\tau}$ as the original function $z_{11}(s)$. By removing the zero shifting section from $z_{11}(s)$, the remainder function $z_{11}^{\prime}(s)$ has zeros at $s=j\,\omega_{\beta 1}$, $s=j\,\omega_{\tau}$ and $s=j\,\omega_{\beta 3}$, which completes the procedure.

Fig. 2.17 shows the network structure for the proposed objective. In that figure, each network elements can be determined by

$$\begin{aligned} C_o &= 1 / d_o, \\ C_2 &= 1 / d_2, \qquad L_2 = d_2 / \omega_{\alpha 2}^2, \\ L_3 &= d_3 \end{aligned} \tag{2-99}$$

where, d_o, d_2 and d_3 are taken from (2-83).

For the sake of brevity, the relationships between the value of ω_{τ} and the control frequencies in the 3-z function are summarized in

Table 2.

Table 2 Relationships between Values of ω_T and Control Frequencies for a 3-z Function

Cases	Values of ω_T	Control Frequencies
1	$0 < \omega_T < \omega_{\beta 1}$	Poles at s=0 s=j $\omega_{\alpha 2}$ s=oo
2	$\omega_{\beta 1} < \omega_T < \omega_{\alpha 1}$	Poles at s=j $\omega_{\alpha 1}$ s=j $\omega_{\alpha 2}$ s=oo
3	$\omega_{\alpha 1} < \omega_T < \omega_{\beta 2}$	Poles at s=0 s=j $\omega_{\alpha 1}$ s=oo
4	$\omega_{\beta 2} < \omega_T < \omega_{\alpha 2}$	Poles at s=0 s=j $\omega_{\alpha 2}$ s=oo
5	$\omega_{\alpha 2} < \omega_T < \omega_{\beta 3}$	Poles at s=0 s=j $\omega_{\alpha 1}$ s=j $\omega_{\alpha 2}$
6	$\omega_{\beta 3} < \omega_T < \infty$	Poles at s=0 s=j $\omega_{\alpha 1}$ s=oo

In Table 2, it is notable that the value of ω_T is also the necessary condition for realizability with the chosen control frequencies. The choice of control frequencies is unique since the procedure with other control frequencies is not realizable. From Table 2, the following observations can be made:

(a) When the value of ω_T is larger than that of the zero to be shifted, all the poles except a lower adjacent pole are chosen for control frequencies.

(b) When the value of ω_T is smaller than that of the zero to be shifted, all the poles except a higher adjacent pole are chosen for control frequencies. The definitions of a lower and higher adjacent pole can be found in Rule 1 or Rule 2 of Art. 2.1.

Procedure for n-z Functions

For a n-z function, the impedance function $z_{11}(s)$ has the form

$$z_{11}(s) = K\frac{(s^2+\omega_{\beta 1}^2)(s^2+\omega_{\beta 2}^2)\ldots\ldots\ldots(s^2+\omega_{\beta n}^2)}{s(s^2+\omega_{\alpha 1}^2)(s^2+\omega_{\alpha 2}^2)\ldots\ldots(s^2+\omega_{\alpha n-1}^2)} \tag{2-100}$$

$z_{11}(s)$ in (2-100) has n zeros and n+1 poles including the extreme frequencies (s=0 and s=oo). From the previous discussion for 2-z or 3-z functions, it is well to emphasize that the number of control frequencies for zero shifting sections is equal to the total number of zeros, that is, one less than the total number of poles. Thus, for a n-z function, n control frequencies are needed for the proposed zero shiftings. Since there are n+1 poles in a n-z function, one of those poles may not be chosen for a control frequency. On the basis of the observations made for a 3-z function case, the rules for choosing control frequencies will be generalized for a n-z case:

(a) For $\frac{\omega_T}{\omega_{\beta 1}} > 1$, choose all the poles except a pole at $s=j\,\omega_{\alpha i-1}$ for control frequencies. (2-101)

(b) For $\frac{\omega_T}{\omega_{\beta i}} < 1$, choose all the poles except a pole at

$s=j\,\omega_{\alpha i}$ for control frequencies. (2-102)

$\omega_{\beta i}$ indicates the frequency value of a zero to be shifted to $s=j\,\omega_{\tau}$. This set of rules can be applied to the poles at the extreme frequencies as well. Rules in (2-101) and (2-102) are restatements of the observations given for 3-z function case and become obvious with reference to the Rules in Art. 2.1. The component functions with proper control frequencies chosen according to two rules in (2-101) and (2-102) belong to three categories and will be assumed to be

$$z_o(s) = \frac{d_o}{s} \tag{2-103}$$

$$z_i(s) = \frac{d_i s}{s^2 + \omega_{\alpha i}^2} \qquad i = 1,2,3\ldots\ldots \tag{2-104}$$

$$z_n(s) = d_n\, s \tag{2-105}$$

Type I section $z_n(s)$ in (2-105) always appears in component functions if the value of ω_τ is larger than $\omega_{\beta n}$ or is smaller than $\omega_{\beta n-1}$. If the value of ω_τ is such that $\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1}$, then a Type II section or $z_o(s)$ may not be taken for a component function. Once component functions are chosen, the next step is to form a matrix equation to determine d_o, d_i and d_n in the component functions. Consider now a matrix equation as

$$\underline{A}\cdot\underline{D} = \underline{B} \tag{2-106}$$

In equation (2-106), $\underline{A}$, $\underline{D}$, and $\underline{B}$ are assumed to be $n\text{x}(n+1)$, $(n+1)\text{x}1$

and nx 1 matrices and will be denoted as

$$\underline{A} = \begin{bmatrix} a_{10} & a_{11} & a_{12} \cdots\cdots\cdots\cdots a_{1n} \\ a_{20} & a_{21} & a_{22} \cdots\cdots\cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n0} & a_{n1} & a_{n2} \cdots\cdots\cdots\cdots a_{nn} \end{bmatrix} \qquad (2\text{-}107)$$

$$\underline{D} = \begin{bmatrix} d_o \\ d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \ldots(2\text{-}108) \qquad \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} \ldots\ldots(2\text{-}109)$$

The subscript a_{ij} of matrix $\underline{A}$ in (2-107) is defined as follows: the upper subscript i indicates the desired new zero, and the lower sub-script j indicates the jth component function. If the kth pole is not chosen for a control frequency, then the column vector $\underline{a_{ik}}$ of $\underline{A}$, which has the lower subscript of k, must be deleted. After deleting one column vector, $\underline{A}$ becomes nxn matrix. A element a_{ij} of matrix $\underline{A}$ is defined as

$$a_{i0} = - \frac{1}{\omega} \bigg|_{\omega = \omega_{\beta i}} \qquad (2\text{-}110)$$

$$a_{ij} = \frac{\omega}{\omega_{\alpha j}^2 - \omega} \bigg|_{\omega = \omega_{\beta i}} \qquad (2\text{-}111)$$

$$a_{in} = \omega \bigg|_{\omega = \omega_{\beta i}} \qquad (2\text{-}112)$$

In case shifting a zero at $s=j\omega_{\beta i}$ to $s=\omega_T$ is required, the row vector $\underline{a_{ij}}$ becomes

$$a_{10} = -\frac{1}{\omega}\Bigg|_{\omega = \omega_\tau} \tag{2-113}$$

$$a_{ij} = \frac{\omega}{\omega^2_{\alpha j} - \omega}\Bigg|_{\omega = \omega_\tau} \tag{2-114}$$

$$a_{in} = \omega\Big|_{\omega = \omega_\tau} \tag{2-115}$$

The column vector $\underline{D}$ is composed of n + 1 elements, one of which must be deleted. If the kth pole is not chosen for a component function, the kth row of D must be deleted.

The column vector $\underline{B}$ is composed of all zero elements except for one element of the value of $X_1(\omega_\tau)$. In case shifting a zero at s=j $\omega_{\beta 1}$ to s=j ω_t is desired, then b_i is defined as $X_1(\omega_\tau)$, and all the other elements are identically zero.

For an illustration, assume that shifting only one zero at s=j $\omega_{\beta 2}$ to s=j ω_τ is desired, and the value of ω_τ satisfies $\omega_{\beta 2} < \omega_\tau < \omega_{\alpha 2}$. Then, according to the rule for choosing control frequencies depicted in (2-101), a pole at s=j $\omega_{\alpha 1}$ may not be chosen for a control frequency. All the other poles in this case are taken for control frequencies. The matrix form to determine component functions is

$$\underline{A}\ \underline{D} = \underline{B}\ ,$$

where $\underline{A}, \underline{D}$ and $\underline{B}$ are defined as

$$\underline{A} = \begin{bmatrix} -\dfrac{1}{\omega_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega^2_{\alpha 2} - \omega^2_{\beta 1}} & \dfrac{\omega_{\beta 1}}{\omega^2_{\alpha 3} - \omega^2_{\beta 1}} & ---- & \dfrac{\omega_{\beta 1}}{\omega^2_{\alpha n-1} - \omega^2_{\beta 1}} & \omega_{\beta 1} \\ -\dfrac{1}{\omega_\tau} & \dfrac{\omega_\tau}{\omega^2_{\alpha 2} - \omega^2_\tau} & \dfrac{\omega_\tau}{\omega^2_{\alpha 3} - \omega^2_\tau} & ---- & \dfrac{}{\omega^2_{\alpha n-1} - \omega^2_\tau} & \omega_\tau \\ ---- & ---- & ---- & ---- & ---- & \\ ---- & ---- & ---- & ---- & ---- & \\ -\dfrac{1}{\omega_{\beta b}} & \dfrac{\omega_{\beta n}}{\omega^2_{\alpha 2} - \omega^2_{\beta n}} & \dfrac{\omega_{\beta n}}{\omega^2_{\alpha 3} - \omega^2_{\beta n}} & ---- & \dfrac{\omega_{\beta n}}{\omega^2_{\alpha n-1} - \omega^2_{\beta n}} & \omega_{\beta n} \end{bmatrix} \tag{2-116}$$

$$\underline{D} = \begin{bmatrix} d_o \\ d_2 \\ d_3 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ X_1(\omega_T) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \qquad \begin{matrix} (2\text{-}117) \\ \\ (2\text{-}118) \end{matrix}$$

Then, the values of $\underline{D}$ are evaluated by

$$d_i = \frac{\Delta_i}{\Delta} \tag{2-119}$$

where, Δ is the determinant of matrix $\underline{A}$ and Δ_i is the subdeterminant of $\underline{A}$. Each d_i obtained from (2-119) defines a component function and the remainder function $z_{11}'(s)$ becomes

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_1(s) - z_3(s) - \ldots\ldots - z_n(s) \tag{2-120}$$

Thus, the $z_{11}'(s)$ in (2-120) has the desired zeros, and the procedure is complete.

This procedure can be justified by induction. The choice of control frequencies in this process is unique, since other choices give unrealizable networks. The steps for this procedure are summarized as

(a) Select appropriate control frequencies from poles of the original function by the rules given in (2-101) and (2-102). The number of control frequencies is n for a n-z function which has n + 1 poles. Thus, one of the poles is not taken for a control frequency.

(b) Formulate a matrix equation in (2-106) and specify each element by considering component functions with proper control frequencies chosen in (a). One of the column vectors in $\underline{A}$, whose lower subscript is the same as the pole that is not chosen for a control frequency, must be deleted. Also, one of the column vectors $\underline{D}$ must be deleted. Then, identify each element of $\underline{B}$ by assigning zero to all elements except the one element of $X_1(\omega_T)$ which appears in the row that ω_T enters.

(c) Evaluate each element of $\underline{D}$ by Cramer's rule from the matrix equation formulated in (b). Realize each component function with known residue d_i and connect the component sections in series.

(d) Obtain the remainder function $z_{11}'(s)$ as in (2-120). This $z_{11}'(s)$ has the proposed zeros.

A predicted network structure for the n-z function with the proposed zero shifting is shown in Fig. 13.

2.3 Proofs of the Conditions for Realizability by Simultaneous Zero Shifting

Techniques in a 2-z Function

The first proof will be given for the condition in (2-38). The necessity of the condition, given in (2-38), can be proven by contradiction as follows: assume that the procedure for Case 1 in 2-z function is realizable if the condition

$$0 < \omega_T < \omega_{\beta 1} \tag{2-121}$$

is violated.

From (2-47) and (2-48), d_o and d_2 become

$$d_o = K\,\frac{\omega_{\beta 2}^2\,(\omega_{\beta 1}^2 - \omega_\tau^2)}{(\omega_{\alpha 1}^2 - \omega_\tau^2)} \qquad (2\text{-}122)$$

$$d_2 = K\,\frac{(\omega_{\beta 1}^2 - \omega_\tau^2)}{(\omega_{\alpha 1}^2 - \omega_\tau^2)} \qquad (2\text{-}123)$$

Since ω_τ was assumed to be positive, only the situation of $\omega_\tau > \omega_{\beta 1}$ is considered.

For $\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1}$, $d_o < 0$ and $d_2 < 0$ (2-124)

For $\omega_\tau > \omega_{\alpha 1}$, $d_o > 0$ and $d_2 > 0$ (2-125)

But, since only passive elements are allowed in this procedure, (2-124) gives unrealizable networks.

On the removal of component functions, the remainder function $z_{11}'(s)$ becomes

$$z_{11}'(s) = z_{11}(s) - z_o(s) - z_2(s)$$

$$= K\,\frac{(s^2+\omega_{\beta 1}^2)(s^2+\omega_{\beta 2}^2)}{s(s^2+\omega_{\alpha 1}^2)} - K\,\frac{\omega_{\beta 2}^2(\omega_{\beta 1}^2-\omega_\tau^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)\,s} - K\,\frac{(\omega_{\beta 1}^2-\omega_\tau^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)}\,s$$

$$= K\,\frac{(\omega_{\alpha 1}^2-\omega_{\beta 1}^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)}\,\frac{(s^2+\omega_\tau^2)(s^2+\omega_{\beta 2}^2)}{s\,(s^2+\omega_{\alpha 1}^2)} \qquad (2\text{-}126)$$

From (2-126), for $\omega_\tau > \omega_{\alpha 1}$,

$$M = K\,\frac{(\omega_{\alpha 1}^2-\omega_{\beta 1}^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)} < 0 \qquad (2\text{-}127)$$

Then, z_{11}' is not positive real and becomes unrealizable.

Therefore, if the condition $0 < \omega_\tau < \omega_{\beta 1}$ is violated, the procedure for Case 1 in 2-z function becomes unrealizable, which contradicts the assumption.

The sufficiency proof may be established by induction.

For a 2-z function, $z_{11}(s)$ has the form

$$z_{11}(s) = K \frac{(s^2+\omega_{\beta 1}^2)(s^2+\omega_{\beta 2}^2)}{s\,(s^2+\omega_{\alpha 1}^2)} \qquad (2\text{-}128)$$

where, $0 < \omega_{\beta 1} < \omega_{\alpha 1} < \omega_{\beta 2}$

Since $X_1(\omega_\tau) = -K \dfrac{(\omega_{\beta 1}^2-\omega_\tau^2)(\omega_{\beta 2}^2-\omega_\tau^2)}{\omega_\tau(\omega_{\alpha 1}^2-\omega_\tau^2)}$, d_o and d_2 in (2-34) and (2-35)

become

$$d_o = K \frac{\omega_{\beta 2}^2(\omega_{\beta 1}^2-\omega_\tau^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)} \qquad (2\text{-}129)$$

$$d_2 = K \frac{(\omega_{\beta 1}^2-\omega_\tau^2)}{(\omega_{\alpha 1}^2-\omega_\tau^2)} \qquad (2\text{-}130)$$

which are the same as in (2-47) and (2-48).

From the two requirements for the successful synthesis by this procedure depicted in (2-21),

$$d_o > 0 \text{ and } d_2 > 0$$

For $d_o > 0$, from (2-129),

$$\omega_\tau < \omega_{\beta 1} \qquad (2\text{-}131)$$

Also for $d_2 > 0$, from (2-130),

$$\omega_\tau < \omega_{\beta 1} \tag{2-132}$$

To make the remainder function positive real,

$$M = K \frac{(\omega_{\alpha 1}^2 - \omega_{\beta 1}^2)}{(\omega_{\alpha 1}^2 - \omega_\tau^2)} > 0 \tag{2-133}$$

From (2-133),

$$\omega_\tau < \omega_{\alpha 1} \tag{2-134}$$

From (2-131), (2-132) and (2-134), the inequality becomes

$$0 < \omega_\tau < \omega_{\beta 1} \tag{2-135}$$

which is the sufficient condition to be realizable by this method for Case 1.

The second proof will be given for the condition in (2-65). The necessity of the condition given in (2-65) also will be proven by contradiction as follows: assume that the procedure for Case 2 in 2-z function is realizable if any inequality of the condition

$$\omega_{\beta 1} < \omega_\tau < \omega_{\alpha 1} \tag{2-136}$$

is violated.

By performing the procedure for Case 2 as in Art. 2.2, d_1 and d_2 are obtained from (2-60) and (2-61) as

$$d_1 = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)}{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2)} \; X_1(\omega_\tau) \tag{2-137}$$

$$d_2 = \frac{(\omega_{\alpha 1}^2 - \omega_\tau^2)}{\omega_\tau(\omega_{\beta 2}^2 - \omega_\tau^2)} \quad X_1(\omega_\tau) \tag{2-138}$$

But since $X_1(\omega_\tau) = - K \dfrac{(\omega_{\beta 1}^2 - \omega_\tau^2)(\omega_{\beta 2}^2 - \omega_\tau^2)}{\omega_\tau(\omega_{\alpha 1}^2 - \omega_\tau^2)}$, d_1 and d_2 in (2-137) and (2-138) become

$$d_1 = K \frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_\tau^2 - \omega_{\beta 1}^2)}{\omega_\tau^2} \tag{2-139}$$

$$d_2 = K \left(1 - \frac{\omega_{\beta 1}^2}{\omega_\tau^2}\right) \tag{2-140}$$

Now, consider two cases that violate the condition in (2-136): one that violates the lower inequality, and the other that violates the upper inequality. If the lower inequality in (2-136) is violated, that is, $\omega_\tau < \omega_{\beta 1}$, d_1 and d_2 in (2-139) and (2-140) have negative values. The component functions in the zero shifting section become unrealizable.

On the removal of zero shifting section, the remainder function has a form

$$z_{11}'(s) = z_{11}(s) - z_1(s) - z_2(s) \tag{2-141}$$

After expanding $z_{11}(s)$ into the form in (2-7) and extracting residues of component functions from the ones of $z_{11}(s)$, equation (2-141) becomes

$$z_{11}'(s) = \frac{k_o}{s} + \frac{(k_1 - d_1)s}{s^2 + \omega_{\alpha 1}^2} + (k_2 - d_2)s \qquad (2\text{-}142)$$

Each residue of $z_{11}'(s)$ in (2-142) must be positive. This requirement comes from the fact that $z_{11}'(s)$ be positive real as was stated in (2-21). To find the residue k_1 of $z_{11}(s)$ at the pole of $s=j\,\omega_{\alpha 1}$ (11),

$$k_1 = \lim_{s^2 \to -\omega_{\alpha 1}^2} \left(\left(\frac{s^2 + \omega_{\alpha 1}^2}{s} \right) z_{11}(s) \right)$$

$$= K \; \frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_{\alpha 1}^2 - \omega_{\beta 1}^2)}{\omega_{\alpha 1}^2} \qquad (2\text{-}143)$$

Also the residue k_2 of $z_{11}(s)$ at the pole of $s=\infty$ becomes

$$k_2 = \lim_{s^2 \to \infty} \left(\left(\frac{1}{s} \right) z_{11}(s) \right)$$

$$= K \qquad (2\text{-}144)$$

If now the upper limit of condition in (2-136) is violated, that is, $\omega_T > \omega_{\alpha 1}$, d_1 in (2-139) becomes

$$d_1 = K\,(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)\left(1 - \frac{\omega_{\beta 1}^2}{\omega_T^2}\right) > K\,(\omega_{\beta 1}^2 - \omega_{\alpha 1}^2)\left(1 - \frac{\omega_{\beta 1}^2}{\omega_{\alpha 1}^2}\right) = K_1, \qquad (2\text{-}145)$$

since $\dfrac{\omega_{\beta 1}^2}{\omega_T^2} < \dfrac{\omega_{\beta 1}^2}{\omega_{\alpha 1}^2}$ for $\omega_T > \omega_{\alpha 1}$.

The inequality in (2-145) violates the requirement for positive realness of $z'_{11}(s)$ and $z'_{11}(s)$ cannot be realizable.

For both cases of the upper or the lower inequality of the condition in (2-136) being violated, this procedure for Case 2 of the 2-z function becomes unrealizable, which contradicts the assumption. Therefore, the necessity of the condition in (2-136) is proven.

The sufficiency proof will be established by induction. Again, two requirements for the successful synthesis by this method are d_1, d_2, k_1 and k_2 obtained from (2-139), (2-140), (2-143) and (2-144) must satisfy the following inequalities:

$$\begin{aligned} &d_1 > 0 , \qquad && d_2 > 0 \\ &d_1 < k_1 , \qquad && d_2 < k_2 \end{aligned} \qquad (2\text{-}145')$$

From (2-139) and (2-140), it can be easily seen that d_1 and d_2 are positive if and only if

$$\omega_\tau > \omega_{\beta 1} \qquad (2\text{-}146)$$

Comparing d_1 in (2-139) and k_1 in (2-143) and utilizing the second requirement in (2-145') yields

$$d_1 < k_1$$

$$K \frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_\tau^2 - \omega_{\beta 1}^2)}{\omega_\tau^2} < K \frac{(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2)(\omega_{\alpha 1}^2 - \omega_{\beta 1}^2)}{\omega_{\alpha 1}^2} \qquad (2\text{-}147)$$

Since $K\,(\omega_{\beta 2}^2 - \omega_{\alpha 1}^2) > 0$, equation (2-147) becomes

$$1 - \frac{\omega_{\beta 1}^2}{\omega_T^2} < 1 - \frac{\omega_{\beta 1}^2}{\omega_{\alpha 1}^2}$$

which requires

$$\omega_T < \omega_{\alpha 1} \tag{2-148}$$

From (2-146) and (2-148), two inequalities can be combined to give

$$\omega_{\beta 1} < \omega_T < \omega_{\alpha 1} \tag{2-149}$$

which makes all the residues satisfy the requirements in (2-145). Thus, the inequality in (2-149) is the sufficient condition for this process.

2.4 Numerical Examples

For the first example, consider the problem of shifting a zero at s=j 1 to s=j 2 and leaving the other the same in the 2-z impedance function

$$z_{11}(s) = \frac{(s^2+1)(s^2+6)}{s(s^2+5)} \tag{2-150}$$

In accordance with the procedure given in Art. 2.2, each of the parameters can be identified as

$$K = 1,\ \omega_{\beta 1}^2 = 1,\ \omega_{\alpha 1}^2 = 5,\ \omega_{\beta 2}^2 = 6 \text{ and } \omega_T^2 = 4 \tag{2-151}$$

The component functions in the zero shifting section are chosen as

$$z_1(s) = \frac{d_1 s}{s^2+5} \tag{2-152}$$

and

$$z_2(s) = d_2 \, s \tag{2-153}$$

The matrix form in (2-58) becomes

$$\begin{bmatrix} \dfrac{2}{5 - 2^2} & 2 \\ \dfrac{6}{5 - 6} & \sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{(1-4)(6-4)}{2(5-4)} \\ 0 \end{bmatrix} \tag{2-154}$$

Evaluating d_1 and d_2 by Cramer's rule gives

$$\begin{aligned} d_1 &= \frac{\Delta_1}{\Delta} = \frac{3}{4} \\ d_2 &= \frac{\Delta_2}{\Delta} = \frac{3}{4} \end{aligned} \tag{2-155}$$

From d_1 and d_2 in (2-155) the component functions are defined to be

$$z_1(s) = \frac{3/4 \, s}{s^2 + 5} \tag{2-156}$$

$$z_2(s) = 3/4 \, s \tag{2-157}$$

The remainder function $z_{11}'(s)$ yields

$$\begin{aligned} z_{11}'(s) &= z_{11}(s) - z_1(s) - z_2(s) \\ &= \frac{1/4 \, (s^2 + 4)(s^2 + 6)}{s \, (s^2 + 5)} \end{aligned} \tag{2-158}$$

The $z_{11}'(s)$ in (2-158) has zeros at $s = j\,2$ and $s = j\sqrt{6}$ that are desired, and the resultant network for this example is shown in Fig. 20.

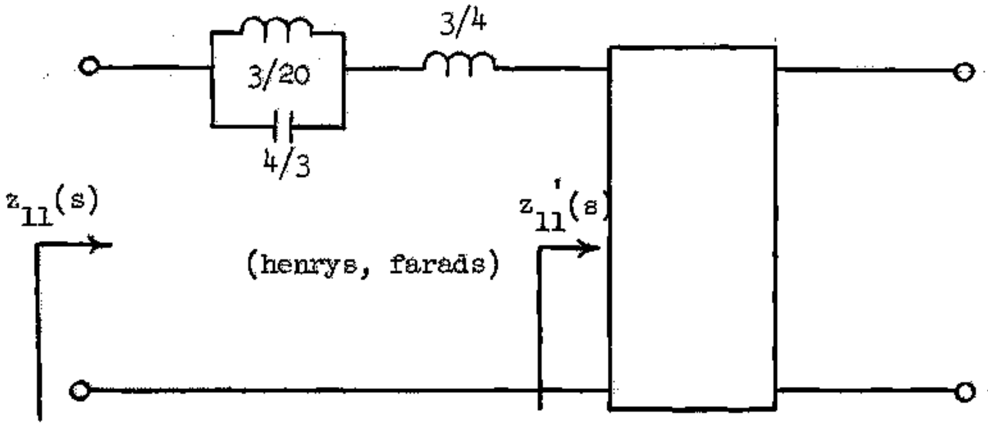

Fig. 20 Realized Network for Example 1

For the second example, consider the problem of shifting a zero at $s=j\sqrt{3}$ to $s=j\ 2$ and leaving all the others the same in a 3-z impedance function

$$z_{11}(s) = \frac{(s^2+1)(s^2+3)(s^2+6)}{s\ (s^2+2)(s^2+5)} \tag{2-159}$$

By the rules for choosing control frequencies in (2-101), the poles at $s=0$, $s=j\sqrt{5}$ and $s=\infty$ are chosen for control frequencies. The component functions are

$$z_o(s) = \frac{d_o}{s} \tag{2-160}$$

$$z_2(s) = \frac{d_2\ s}{s^2+5} \tag{2-161}$$

$$z_3(s) = d_3\ s \tag{2-162}$$

The matrix form in (2-81) can be used to determine the values of d_o, d_2 and d_3 in (2-160), (2-161) and (2-162).

$$\begin{bmatrix} -\frac{1}{1} & \frac{1}{5-1} & 1 \\ -\frac{1}{2} & \frac{2}{5-4} & 2 \\ -\frac{1}{6} & \frac{6}{5-6} & 6 \end{bmatrix} \cdot \begin{pmatrix} d_o \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \\ 0 \end{pmatrix} \tag{2-163}$$

Evaluating the determinant and subdeterminants of (2-163) gives

$$\Delta = -25\sqrt{6}/8 \qquad \Delta_1 = -15\sqrt{6}/8$$

$$\Delta_2 = -5\sqrt{6}/4 \qquad \Delta_3 = -25\sqrt{6}/16,$$

and d_o, d_2 and d_3 become

$$d_o = \frac{\Delta_1}{\Delta} = \frac{3}{5}$$

$$d_2 = \frac{\Delta_2}{\Delta} = \frac{2}{5}$$

$$d_3 = \frac{\Delta_3}{\Delta} = \frac{1}{2} \tag{2-164}$$

The residues d_o, d_2 and d_3 in (2-164), in turn, determine the component functions as

$$z_o(s) = \frac{3/5}{s} \tag{2-165}$$

$$z_2(s) = \frac{2/5\ s}{s^2 + 5} \tag{2-166}$$

$$z_3(s) = 1/2\ s \tag{2-167}$$

The remainder function $z_{11}^{'}(s)$ in this case will have the form

$$z_{11}^{'}(s) = z_{11}(s) - z_o(s) - z_2(s) - z_3(s)$$

$$= \frac{1/2\ (s^2+1)(s^2+4)(s^2+6)}{s\ (s^2+2)(s^2+5)} \qquad (2\text{-}168)$$

which has zeros at s=j1, s=j2 and $s=j\sqrt{6}$.

The component functions obtained by this procedure can be realized as in Fig. 21.

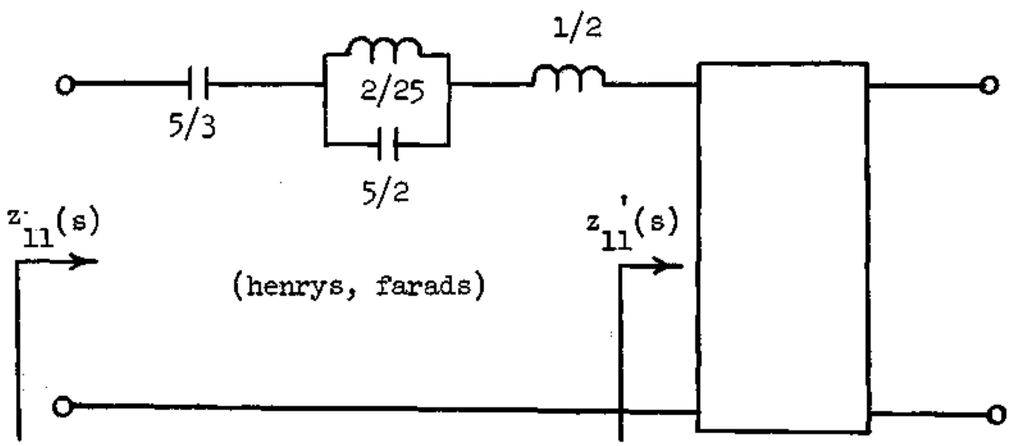

Fig. 21 Realized Network for Example 2

CHAPTER III

A METHOD FOR SHIFTING i ZEROS OF A REACTANCE FUNCTION TO DESIRED LOCATIONS AND LEAVING THE OTHERS THE SAME

The problem that will be considered is to find a method to shift i zeros to proper locations and leave the others the same in a reactance function. Because of the complexity, a method to shift two zeros to the desired locations ($s=j\omega_{\tau i}$) in a 2-z function will be considered in the beginning. Generalization of this method in a n-z function will be investigated, but this requires repetitive use of the procedures outlined in Chapter II.

3.1 Procedure for 2-z Functions

The problem that is considered here is shown in Fig. 22.

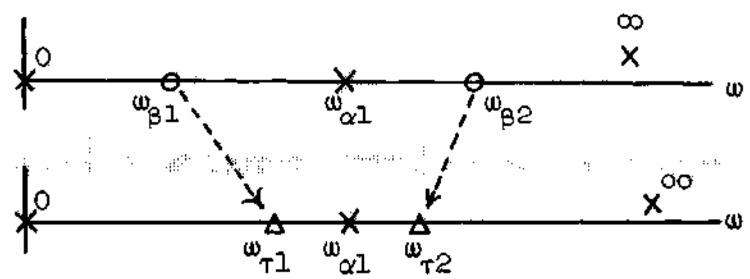

Fig. 22 Original and New Zero-Pole Distributions

In Fig. 22, the zero at $s=j\omega_{\beta 1}$ and/or the one at $s=j\omega_{\beta 2}$ is shifted to $s=j\omega_{\tau 1}$ and/or $s=j\omega_{\tau 2}$, respectively. It is assumed that frequency values of new zeros satisfy the following inequalities:

$$\omega_{\beta 1} < \omega_{\tau 1} < \omega_{\alpha 1}$$

$$\omega_{\alpha 1} < \omega_{\tau 2} < \omega_{\beta 2} \qquad (3\text{-}1)$$

There are three possible choices of control frequencies in this case; poles at $s=0$, $s=j\omega_{\alpha 1}$ or poles at $s=j\omega_{\alpha 1}$, $s=\infty$ or poles at $s=0$, $s=j\omega_{\alpha 1}$, $s=\infty$. Two control frequencies are to be chosen. However, the limitations on $\omega_{\tau i}$ become more strict than in the single zero shift case. On the other hand, the choice of three control frequencies eliminates the limitations on the value of $\omega_{\tau i}$, but repetitive use of the single zero shifting (discussed in Chapter II) is required.

First, consider the case of choosing poles at $s=j\omega_{\alpha 1}$ and $s=\infty$ for control frequencies. Then, the component functions will have the forms

$$z_1(s) = \frac{d_1\ s}{s^2 + \omega_{\alpha 1}^2} \tag{3-2}$$

$$z_2(s) = d_2\ s \tag{3-3}$$

Removal of the component function $z_1(s)$ in (3-2) shifts the zero at $s=j\omega_{\beta 1}$ away from the origin and the zero at $s=j\omega_{\beta 2}$ toward the origin. This aids the desired shift of zeros. The component function $z_2(s)$ in (3-3), however, produces a Type I section, which shifts all the zeros away from the origin. Thus, removal of $z_2(s)$ in (3-3) aids the shifting of the zero at $s=j\omega_{\beta 1}$ to $s=j\omega_{\tau 1}$ but prevents the shift of the zero at $s=j\omega_{\beta 2}$ to $s=j\omega_{\tau 2}$. If the residue d_2 in $z_2(s)$ is small, then the proposed zero shifting may be achieved by these control frequencies. By analogy with the procedure given in Chapter II, the remainder function $z_{11}(s)$, after removing zero shifting sections, must satisfy the following relations.

$$z_{11}'(j\omega_{\tau 1}) = 0$$

$$z_{11}'(j\omega_{\tau 2}) = 0 \qquad (3\text{-}4)$$

The conditions in (3-4) are imposed because the $z_{11}'(s)$ must have zeros at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$. But, since the remainder function can be obtained by extracting component functions from the original function $z_{11}(s)$, equations in (3-4) becomes

$$z_{11}(s) - z_1(s) - z_2(s) \Big|_{s=j\omega_{\tau 1}} = 0 \qquad (3\text{-}5)$$

$$z_{11}(s) - z_1(s) - z_2(s) \Big|_{s=j\omega_{\tau 2}} = 0 \qquad (3\text{-}6)$$

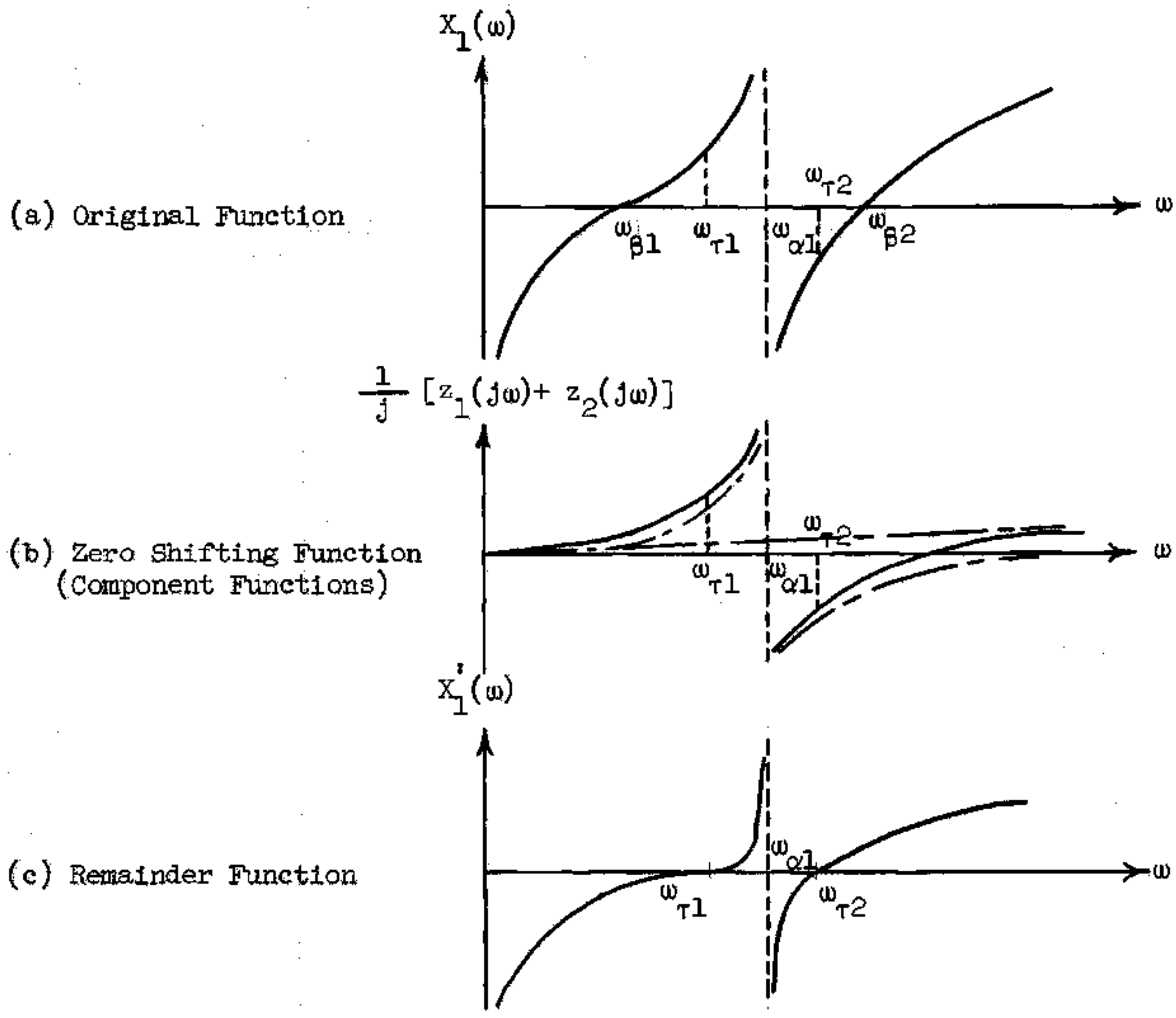

Fig. 23 Graphical Interpretation of the Procedure for a 2-z Function

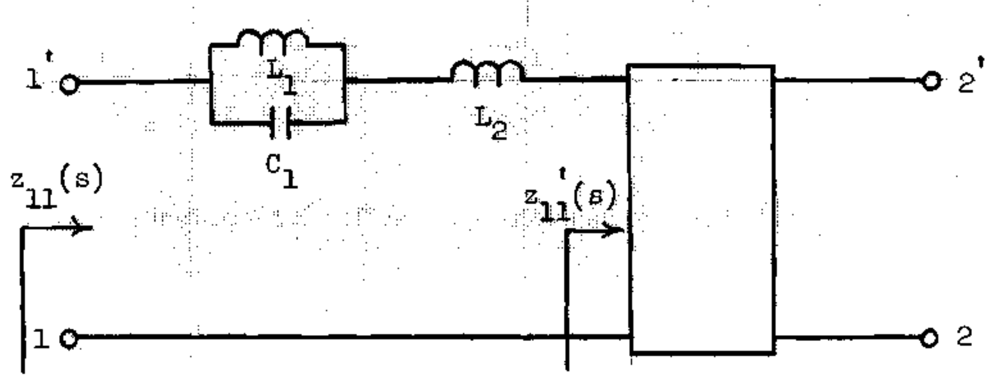

Fig. 24 Network Arrangement for a 2-z Function

Substituting control frequencies in (3-2) and (3-3) into (3-5) and (3-6) yields

$$z_{11}(j\omega_{\tau 1}) - \frac{j\, d_1\, \omega_{\tau 1}}{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2} - j\, d_2 \omega_{\tau 1} = 0 \qquad (3\text{-}7)$$

$$z_{11}(j\omega_{\tau 2}) - \frac{j\, d_1\, \omega_{\tau 2}}{\omega_{\alpha 1}^2 - \omega_{\tau 2}^2} - j\, d_2 \omega_{\tau 2} = 0 \qquad (3\text{-}8)$$

Since $z_{11}(j\omega) = j\, X_1(\omega)$, equations (3-7) and (3-8) can be put into the matrix form as

$$\begin{bmatrix} \dfrac{\omega_{\tau 1}}{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2} & \omega_{\tau 1} \\ \\ \dfrac{\omega_{\tau 2}}{\omega_{\alpha 1}^2 - \omega_{\tau 2}^2} & \omega_{\tau 2} \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ \\ d_2 \end{bmatrix} = \begin{bmatrix} X_1(\omega_{\tau 1}) \\ \\ X_2(\omega_{\tau 2}) \end{bmatrix} \qquad (3\text{-}9)$$

Evaluating d_1 and d_2 in (3-9) by Cramer's rule gives

$$d_1 = \frac{(\omega_{\alpha 1}^2 - \omega_{\tau 1}^2)(\omega_{\tau 2}^2 - \omega_{\alpha 1}^2)}{\omega_{\tau 1}\omega_{\tau 2}(\omega_{\tau 2}^2 - \omega_{\tau 1}^2)} \left[\omega_{\tau 2}\, X_1(\omega_{\tau 1}) - \omega_{\tau 1} X_1(\omega_{\tau 2}) \right] \qquad (3\text{-}10)$$

$$d_2 = \frac{X_1(\omega_{\tau 2})(\omega_{\tau 2}^2 - \omega_{\alpha 1}^2)}{\omega_{\tau 2}(\omega_{\tau 2}^2 - \omega_{\tau 1}^2)} + \frac{X_1(\omega_{\tau 1})(\omega_{\alpha 1}^2 - \omega_{\tau 1}^2)}{\omega_{\tau 1}(\omega_{\tau 2}^2 - \omega_{\tau 1}^2)} \qquad (3\text{-}11)$$

The remainder function is defined by known component functions as

$$z_{11}'(s) = z_{11}(s) - z_1(s) - z_2(s) \qquad (3\text{-}12)$$

which has proper zeros at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$.

In order for this procedure to be realizable, d_1 and d_2 in (3-10) and (3-11) must be positive. The residue d_1 in (3-10) is positive, since the values of $\omega_{\tau i}$ are assumed to be constrained as shown in (3-1). But the equation (3-11) requires more constraints on the values of $\omega_{\tau i}$ in order to make d_2 positive, that is,

$$-\frac{X_1(\omega_{\tau 2})}{X_1(\omega_{\tau 1})} < \left(\frac{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2}{\omega_{\tau 2}^2 - \omega_{\tau 1}^2}\right) \cdot \frac{\omega_{\tau 2}}{\omega_{\tau 1}} \tag{3-13}$$

Therefore, necessary conditons by realizability by the procedure with control frequencies at $s=j\,\omega_{\alpha 1}$ and $s=\infty$ for the proposed zero shifting as illustrated in Fig. 22 are

$$\omega_{\beta 1} < \omega_{\tau 1} < \omega_{\alpha 1},$$

$$\omega_{\alpha 1} < \omega_{\tau 2} < \omega_{\beta 2}$$

and
$$-\frac{X_1(\omega_{\tau 2})}{X_1(\omega_{\tau 1})} < \left(\frac{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2}{\omega_{\tau 2}^2 - \omega_{\tau 1}^2}\right) \cdot \frac{\omega_{\tau 2}}{\omega_{\tau 1}} \tag{3-14}$$

Fig. 23 shows the behaviors of the original function, component function and remainder function. In that figure, the impedance levels of the zero shifting function at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$ are made to be the same as the ones of original function at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$. By removing the zero shifting function which is composed of component functions, the impedance levels at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$ become identically zero.

Secondly, three control frequencies are chosen to improve the limitations for realizability. If all of the poles of $z_{11}(s)$ are chosen for control frequencies, the component functions are defined as

$$z_o(s) = \frac{d_o}{s} \tag{3-15}$$

$$z_1(s) = \frac{d_1 s}{s^2 + \omega_{\alpha 1}^2} \tag{3-16}$$

$$z_2(s) = d_2 s \tag{3-17}$$

Since only two known conditions are available as in (3-4), there is no way to specify all three component functions simultaneously. But, the repetitive use of the procedure for shifting one zero to a desired location and leaving the others the same can determine the residues of all three component functions. This involves two steps:

(a) Consider the problem of shifting a zero at $s=j\omega_{\beta 1}$ to $s=j\omega_{\tau 1}$ and leaving the other the same. Perform the procedure discussed in Chapter II. The remainder function in this step will have zeros at $s=j\omega_{\tau 1}$ and $s=j\omega_{\beta 2}$.

(b) Apply again the procedure to shift the zero at $s=j\omega_{\beta 2}$ to $s=j\omega_{\tau 2}$ and leave the other zero the same. This step gives the remainder function which has zeros at $s=j\omega_{\tau 1}$ and $s=j\omega_{\tau 2}$ and the proposed zero shifting is achieved.

The first step involves the same procedure as one for Case 2 in Chapter II. However, the second step involves the same procedure as one for Case 3 in Chapter II and also requires a knowledge of the remainder function resulting from the first step. Each step requires two component functions. Decomposition of $z_1(s)$ in (3-16) into two parts yields

$$z_1(s) = z_{1a}(s) + z_{1b}(s)$$

$$= \frac{d_{1a}s}{s^2 + \omega_{\alpha 1}^2} + \frac{d_{1b}s}{s^2 + \omega_{\alpha 1}^2} \tag{3-18}$$

The first step is to find the values of d_{1a} and d_2 in (3-18) and (3-17). For the sake of brevity, the matrix equation in (2-58) will be given without discussion. The detailed explanation was given in Chapter II. For $\omega_{\beta 1} < \omega_{\tau 1} < \omega_{\alpha 1}$, the matrix equation becomes

$$\begin{bmatrix} \frac{1}{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2} & \omega_{\tau 1} \\ \frac{1}{\omega_{\alpha 1}^2 - \omega_{\beta 2}^2} & \omega_{\beta 2} \end{bmatrix} \cdot \begin{bmatrix} d_{1a} \\ d_2 \end{bmatrix} = \begin{bmatrix} X_1(\omega_\tau) \\ 0 \end{bmatrix} \tag{3-19}$$

The next step is to find d_o and d_{1b} in (3-15) and (3-18). Let the remainder function $z_{11}'(s)$ be defined as

$$z_{11}'(s) = z_{11}(s) - z_{1a}(s) - z_2(s)$$

$$\text{and} \qquad z_{11}'(j\omega) = j\, X_1'(\omega) \tag{3-20}$$

By the similar procedure as the first step, the matrix equation to determine the values of d_o and d_{1b} becomes

$$\begin{bmatrix} -\frac{1}{\omega_{\tau 1}} & \frac{1}{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2} \\ -\frac{1}{\omega_{\tau 2}} & \frac{1}{\omega_{\alpha 1}^2 - \omega_{\tau 2}^2} \end{bmatrix} \cdot \begin{bmatrix} d_o \\ d_{1b} \end{bmatrix} = \begin{bmatrix} 0 \\ X_1'(\omega_{\tau 2}) \end{bmatrix} \tag{3-21}$$

The matrix equations (3-19) and (3-21) determine d_o, d_{1a}, d_{1b} and d_2 which in turn specify the component functions. After finding the values of d_{1a} and d_{1b}, they are combined to be realized by one component network.

$$d_1 = d_{1a} + d_{1b} \tag{3-22}$$

For the procedure with three control frequencies, there is no additional limitation on the value of ω_τ other than (3-1). Since this procedure is simply a repetitive use of the one in Chapter II, each step has no limitation on the values of ω_τ other than the assumptions of its own and therefore, this whole procedure will have no limitations. Thus, the method to shift two zeros to proper locations for 2-z function is always possible with three control frequencies provided that the proper locations are not the same as the poles of itself.

The network structure for this procedure is shown in Fig. 25.

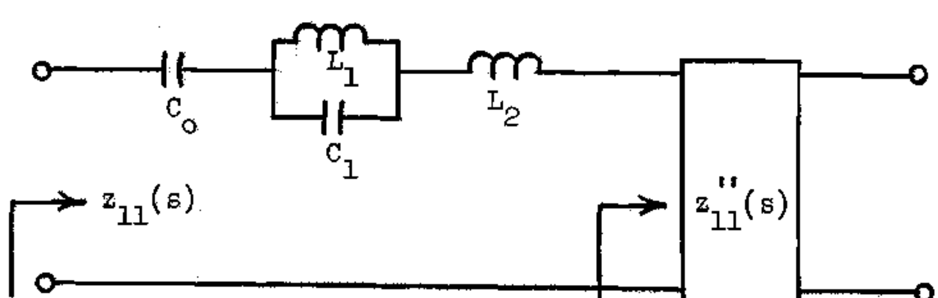

Fig. 25 Network Arrangement for a 2-z Function Realized by the Procedure with Three Control Frequencies

The procedure with control frequencies at s=0 and $s=j\omega_{\alpha 1}$ can be found in Table 3; it requires more strict realizability constraints on the values of $\omega_{\tau i}$ than the one in (3-14). The procedure with control frequencies at s=0 and s=oo becomes unrealizable, since it requires negative residues of component functions.

Table 3 Relationships Between Control Frequencies and Conditions for Realizability in a 2-z Function

Control Frequencies	Conditions for Realizability
Poles at s=0 s=oo	Unrealizable
Poles at s=0 $s=j\omega_{\alpha 1}$	$\omega_{\beta 1}<\omega_{\tau 1}<\omega_{\alpha 1},\ \omega_{\alpha 1}<\omega_{\tau 2}<\omega_{\beta 2}$ $-\dfrac{X_1(\omega_{\tau 2})}{X_1(\omega_{\tau 1})}<\dfrac{\omega_{\tau 2}^2-\omega_{\alpha 1}^2}{\omega_{\alpha 1}^2-\omega_{\tau 1}^2}$
Poles at $s=j\omega_{\alpha 1}$ s=oo	$\omega_{\beta 1}<\omega_{\tau 1}<\omega_{\alpha 1},\ \omega_{\alpha 1}<\omega_{\tau 2}<\omega_{\beta 2},$ $-\dfrac{X_1(\omega_{\tau 2})}{X_1(\omega_{\tau 1})}<\left(\dfrac{\omega_{\alpha 1}^2-\omega_{\tau 1}^2}{\omega_{\tau 2}^2-\omega_{\alpha 1}^2}\right)\cdot\dfrac{\omega_{\tau 2}}{\omega_{\tau 1}}$
Poles at s=0 $s=j\omega_{\alpha 1}$ s=oo	$\omega_{\beta 1}<\omega_{\tau 1}<\omega_{\alpha 1},$ $\omega_{\alpha 1}<\omega_{\tau 2}<\omega_{\beta 2}$

3.2 Procedure for n-z Functions

The problem that is considered here is to shift i zeros to desired locations and leave the others the same. To avoid the limitations on the values of $\omega_{\tau i}$ for realizability, the procedure given in Chapter II is made use of repeatedly for i times. In general, all the pole frequencies are chosen for control frequencies and thus n + 1 component functions are needed for the procedure in a n-z function. Then, i matrix equations are

$$\begin{aligned} \underline{A_1}\ \underline{D_1} &= \underline{B_1} \\ \underline{A_2}\ \underline{D_2} &= \underline{B_2} \\ &\cdots\cdots\cdots\cdots \\ \underline{A_i}\ \underline{D_i} &= \underline{B_i} \end{aligned} \tag{3-23}$$

In each step, the remainder function must be obtained in order to initiate the next step. Each d_j determined from (3-23) is a partial or whole residue of a component function. If d_j is a part of the residue, then all the d_j's are combined to get a whole residue of each component function. The remainder function in the ith step becomes

$$z_{11}^{i}(s) = z_{11}(s) - (\text{sum of component functions}) \tag{3-24}$$

which has the proposed zero-pole configuration along the positive frequency axis. The procedure for an n-z function can be summarized as following:

(a) Choose any zero at $s=j\omega_{\beta j}$ to be shifted to $s=j\omega_{\tau j}$ and perform the procedure to shift the zero at $s=j\omega_{\beta j}$ to $s=j\omega_{\tau j}$ and leave the others the same as in Chapter II. Formulate a matrix equation to determine component functions z_{oa}, z_{1a}, z_{2a}, ..., and solve for each residue of the component function. Define a remainder function by the equation as in (3-24).

(b) Choose the next zero at $s=j\omega_{\beta k}$ to be shifted to $s=j\omega_{\tau k}$. Proceed with the same procedure as in step (a).

(c) Repeat step (b) until all the zeros are shifted to the proper locations.

In general this procedure is tedious but realizability is guaranteed since each step is always realizable provided that the value of $\omega_{\tau i}$ is not the same as that of a pole in the n-z function.

3.3 Numerical Examples

Consider the problem of shifting a zero at $s=j1$ to $s=j2$ and the other zero at $s=j\sqrt{10}$ to $s=j3$ in a 2-z function

$$z_{11}(s) = \frac{(s^2+1)(s^2+10)}{s\,(s^2+5)} \tag{3-25}$$

Two methods given in Table 3 are available for the proposed zero shifting.

First, the procedure with control frequencies at $s=j\sqrt{5}$ and $s=\infty$ is discussed. All the parameters, in this case, are identified as

$$\omega_{\beta 1} = 1\,, \qquad \omega_{\beta 2} = \sqrt{10}\,, \qquad \omega_{\alpha 1} = \sqrt{5}\,,$$

$$\omega_{\tau 1} = 2, \qquad \omega_{\tau 2} = 3 \tag{3-26}$$

To see if the conditions in Table 3 are fulfilled,

$$X_1(\omega_{\tau 1}) = 9$$

$$X_2(\omega_{\tau 2}) = -\,2/3$$

and
$$-\,X_1(\omega_{\tau 2})/X_1(\omega_{\tau 1}) = 2/27 \tag{3-27}$$

but
$$\left(\frac{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2}{\omega_{\tau 2}^2 - \omega_{\alpha 1}^2}\right)\cdot\frac{\omega_{\tau 2}}{\omega_{\tau 1}} = \frac{3}{4} \tag{3-28}$$

Comparing (3-27) with (3-28) gives

$$-\,\frac{X_1(\omega_{\tau 2})}{X_1(\omega_{\tau 1})} < \left(\frac{\omega_{\alpha 1}^2 - \omega_{\tau 1}^2}{\omega_{\tau 2}^2 - \omega_{\alpha 1}^2}\right)\cdot\frac{\omega_{\tau 2}}{\omega_{\tau 1}} \tag{3-29}$$

which satisfies the condition for realizability with control frequencies at $s=j\,5$ and ∞ in Table 3.

The component functions with control frequencies at $s=j\sqrt{5}$ and $s=\infty$ are

$$z_1(s) = \frac{d_1\, s}{s^2 + 5} \tag{3-30}$$

$$z_2(s) = d_2\, s \tag{3-31}$$

The matrix equation in (3-9) can be utilized to find the values of d_1 and d_2 in (3-30) and (3-31). Substituting each parameter values in (3-26) into (3-9) yields

$$\begin{bmatrix} \frac{2}{5-4} & 2 \\ \frac{3}{5-9} & 3 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -\frac{2}{3} \end{bmatrix} \tag{3-32}$$

Evaluating (3-32) for d_1 and d_2 gives

$$d_1 = 34/9 \tag{3-33}$$

and

$$d_2 = 13/18 \tag{3-34}$$

which define the component function in (3-30) and (3-31) as

$$z_1(s) = \frac{34/9\ s}{s^2 + 5} \tag{3-35}$$

$$z_2(s) = 13/18\ s \tag{3-36}$$

The remainder function $Z_{11}'(s)$ becomes

$$z_{11}'(s) = z_{11}(s) - z_1(s) - z_2(s)$$

$$= \frac{5}{18} \frac{(s^2+4)(s^2+9)}{s\ (s^2+5)} \tag{3-37}$$

which has the proposed zeros.

The network structure realized by the procedure with control frequencies at $s=j\sqrt{5}$ and $s=\infty$ is shown in Fig. 26(a).

Secondly, by the procedure with three control frequencies, the component functions are

$$z_o(s) = \frac{d_o}{s} \tag{3-38}$$

$$z_1(s) = \frac{d_1 \ s}{s^2 + 5} \tag{3-39}$$

and

$$z_2(s) = d_2 \ s \tag{3-40}$$

Decomposition of $z_1(s)$ yields

$$\begin{aligned} z_1(s) &= z_{1a}(s) + z_{1b}(s) \\ &= \frac{d_{1a}s}{s^2 + 5} + \frac{d_{1b}s}{s^2 + 5} \end{aligned} \tag{3-41}$$

To determine the values of d_o, d_{1a}, d_{1b} and d_2, two matrix equations are required.

(Step 1) Consider the problem of shifting a zero at s=j1 to s=j2 and leaving the other zero unchanged. The same matrix equation as in (3-19) can be applied in this case. Substituting all the parameter values into (3-19) gives

$$\begin{bmatrix} \frac{2}{5 - 4} & 2 \\ \frac{\sqrt{10}}{5 - 10} & \sqrt{10} \end{bmatrix} \cdot \begin{bmatrix} d_{1a} \\ d_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix} \tag{3-42}$$

which leads to the solutions for d_{1a} and d_2 as

$$d_{1a} = 15/4 \tag{3-43}$$

and $$d_2 = 3/4 \tag{3-44}$$

The remainder function $z_{11}^{'}(s)$ becomes

$$z_{11}^{'}(s) = z_{11}(s) - z_{1a}(s) - z_2(s)$$

$$= \frac{1}{4} \; \frac{(s^2+4)(s^2+10)}{s\,(s^2+5)} \tag{3-45}$$

(Step 2) Consider now the problem of shifting a zero at $s=j\sqrt{10}$ to $s=j3$ and retaining the other zero at $s=j2$. The same matrix equation as in (3-21) can be applied in this case. Evaluating each matrix elements in (3-21) gives

$$\begin{pmatrix} -\frac{1}{2} & \frac{2}{5-4} \\ -\frac{1}{3} & \frac{3}{5-9} \end{pmatrix} \begin{pmatrix} d_o \\ d_{1b} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{5}{48} \end{pmatrix} \tag{3-46}$$

Solving (3-46) for d_o and d_{1b} yields

$$d_o = 1/5 \tag{3-47}$$

$$d_{1b} = 1/20 \tag{3-48}$$

The remainder function for step 2 becomes

$$z_{11}^{''}(s) = z_{11}^{'}(s) - z_o(s) - z_{1b}(s)$$

$$= \frac{1}{4} \; \frac{(s^2+4)(s^2+9)}{s(s^2+5)} \tag{3-49}$$

which completes the procedure.

The parameters d_{1a} in (3-43) and d_{1b} in (3-48) are added and can be realized from one component network as

$$d_1 = d_{1a} + d_{1b}$$
$$= 19/5 \qquad (3\text{-}50)$$

The residues d_o, d_1 and d_2 in (3-47), (3-50) and (3-44) define the component functions and the network structure for this procedure is shown in Fig. 26(b).

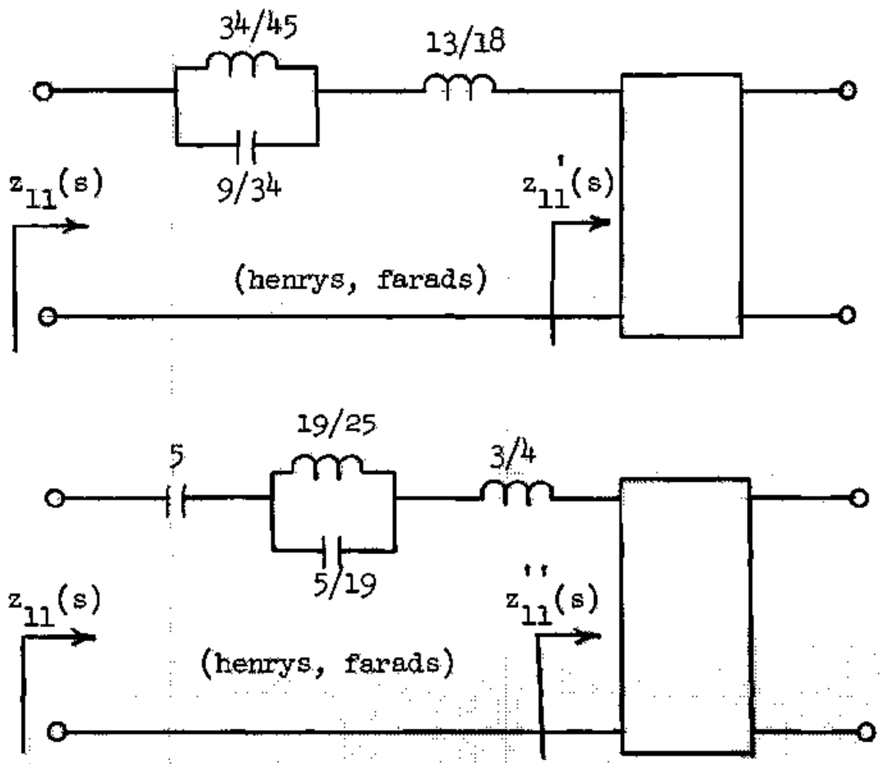

Fig. 26 Two Realized Networks for Example (a) by the Procedure with Two Control Frequencies (b) by the Procedure with Three Control Frequencies.

CHAPTER IV

APPLICATIONS TO THE SYNTHESIS OF REACTANCE NETWORKS

The synthesis of network structures, utilizing the techniques described in Chapter II and Chapter III, will now be discussed. These structures will be made up of two sections; one is a zero shifting section, and the other is a zero producing section. These additional approaches allow certain transfer functions to be synthesized, which cannot be realized with the basic ladder synthesis technique using only Type I and/or Type II and/or Type III basic sections. This synthesis method, like zero shifting methods, is based on the following two conditions(9):

(a) In a purely reactive ladder network, the poles of $z_{12}(s)$ are simple. Moreover, $z_{11}(s)$ and $z_{12}(s)$ possess these same poles.

(b) Every zero of $z_{12}(s)$ of a reactive ladder network must be a zero of an impedance function of shunt element (this is an imposed condition). A set of parameters ($z_{11}(s)$ and $z_{12}(s)$, $y_{22}(s)$ and $y_{12}(s)$) is assumed to be known from (1-1) or (1-2) in Chapter I and also are assumed to satisfy the physically realizable conditions. No attempt will be made to maximize the gain of the transfer function nor to minimize the number of elements of the synthesized network.

4.1 Modified Zero Shifting Synthesis

The network for the modified zero shifting synthesis method is composed of two main sections as shown in Fig. 27.

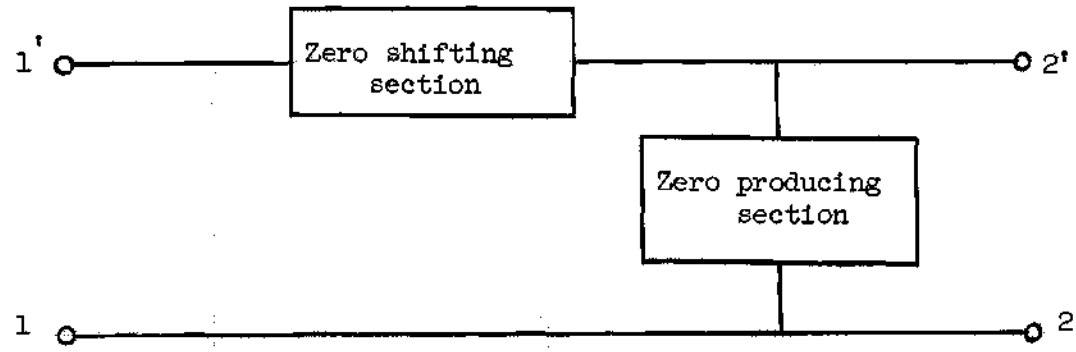

Fig. 27 Network Arrangement for Modified Zero Shifting Synthesis

In Fig. 27, the zero shifting section is shown as a clear box, and the zero producing section is shown as a shaded box. The series section, which shifts the zeros of $z_{11}(s)$ to desired locations, is composed of n-1 or n series branches, where n denotes the number of zeros of $z_{11}(s)$. For instance, if the transfer function $z_{12}(s)$ has n-i zeros that are identical with the driving point function $z_{11}(s)$, while the other i zeros are different from $z_{11}(s)$, then the procedure described in Chapter III can be used to realize the zero shifting section. The remainder function obtained by removing the zero shifting section must have proper zeros of the impedance function that are identical with transmission zeros. Zeros of $z_{12}(s)$ are defined as transmission zeros. The zero producing section can be realized by classical Foster's or Cauer's synthesis. One of the procedures for realizing the zero producing section is as follows:

Step 1. Take the reciprocal of the remainder function, that is, obtain an admittance function y_{11}' as

$$y_{11}' = 1/z_{11}' \tag{4-1}$$

Step 2. Expand y_{11}' into partial fractions as

$$y_{11}'(s) = \frac{a_1 s}{s^2 + \omega_\tau^2} + \frac{a_2 s}{s^2 + \omega_{\tau 2}^2} + \cdots\cdots + \frac{a_i s}{s^2 + \omega_{\tau i}^2} \tag{4-2}$$

and realize each component function as a shunt arm shown in Fig. 28.

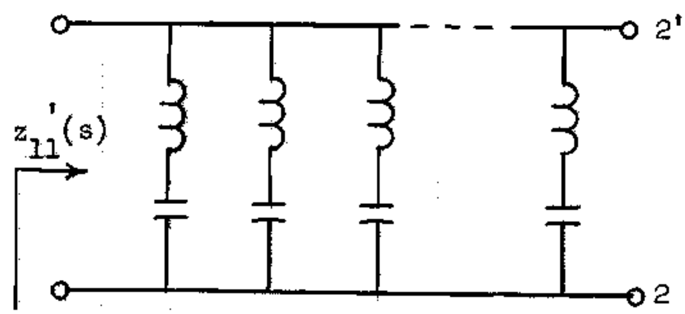

Fig. 28 Zero Producing Section

The pole of each shunt arm admittance function will be devoted to producing a zero of $z_{12}(s)$.

A similar procedure of synthesis holds for $y_{22}(s)$ and $y_{12}(s)$ with the same network arrangement as in Fig. 27.

As an illustration, consider the problem of synthesizing

$$z_{11}(s) = \frac{(s^2 + 1)(s^2 + 10)}{s(s^2 + 5)} \tag{4-3}$$

$$z_{12}(s) = \frac{(s^2 + 4)(s^2 + 9)}{s(s^2 + 5)} \tag{4-4}$$

The zero shifting section in this case is responsible for shifting a zero at $s=j1$ to $s=j2$ and the other zero at $s=j\sqrt{10}$ to $s=j3$. The procedure to realize the zero shifting section was discussed in Art. 3.3.

From (3-37), the remainder function is

$$z_{11}'(s) = \frac{5}{18} \; \frac{(s^2+4)(s^2+9)}{s(s^2+5)} \tag{4-5}$$

To synthesize the zero producing section in (4-5),

$$y_{11}'(s) = \frac{18}{5} \; \frac{s(s^2+5)}{(s^2+4)(s^2+9)} = \frac{a_1 s}{s^2+4} + \frac{a_2 s}{s^2+9} \tag{4-6}$$

where,

$$a_1 = \lim_{s^2 \to -4} \left(\frac{s^2+4}{s} \quad y_{11}'(s) \right) = \frac{18}{25} \tag{4-7}$$

$$a_2 = \lim_{s^2 \to -4} \left(\frac{s^2+9}{s} \quad y_{11}'(s) \right) = \frac{72}{25} \tag{4-8}$$

Therefore, the realized reactive network, within a constant multiplier, has the form as in Fig. 29.

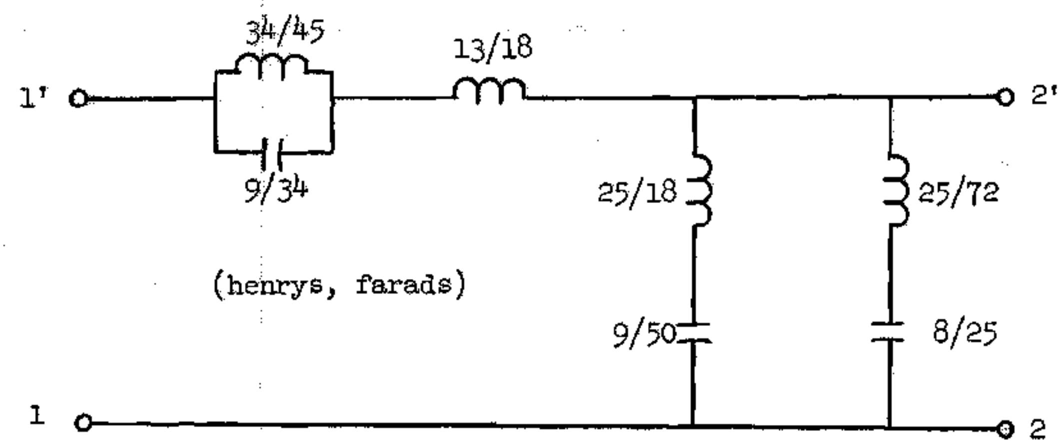

Fig. 29 Realized LC Network for Reactance Functions (4-3) and (4-4)

4.2 Parallel-Ladder Synthesis (3,7)

On an admittance basis, the susceptive parameters can be found by adding each ladder network function, when j ladder networks are properly connected in parallel as shown in Fig. 30.

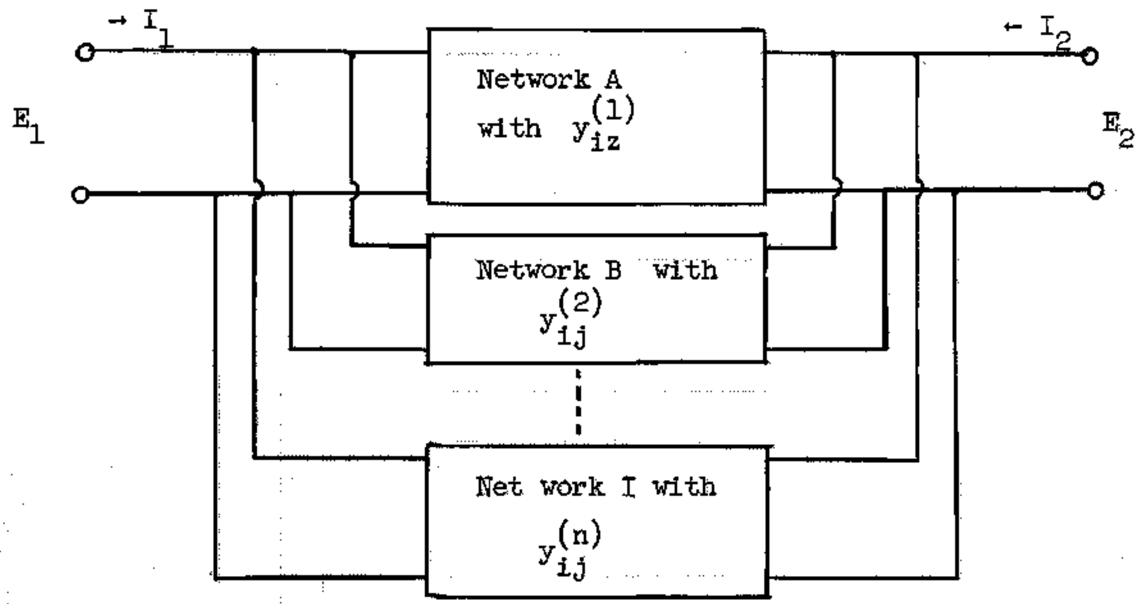

Fig. 30 Parallel-Ladder Network

Thus, if y_{ij} is the composite network parameter and $y_{ij}^{(k)}$ for k=1,2, 3,..., are the component network parameters, then

$$y_{ij} = \sum_{k=1}^{n} y_{ij}^{(k)}$$

$$= y_{ij}^{(1)} + y_{ij}^{(2)} + y_{ij}^{(3)} + \ldots\ldots + y_{ij}^{(n)} \qquad (4\text{-}9)$$

For the lossless network, the transfer admittance can be written as

$$-y_{12}(s) = \frac{P(s)}{Q(s)} = \frac{a_o + a_1 s^2 + a_2 s^4 + \ldots\ldots + a_m s^{2m}}{s(s^2+\omega_{\alpha 1}^2)(s^2+\omega_{\alpha 2}^2 \ldots\ldots (s^2+\omega_{\alpha m-1}^2)} \quad (4\text{-}10)$$

It is assumed that $y_{22}(s)$ is in factored form as

$$y_{22}(s) = M\frac{A(s)}{Q(s)}$$

$$= M \frac{(s^2+\omega_{\beta 1}^2)(s^2+\omega_{\beta 2}^2)(s^2+\omega_{\beta 3}^2)\ldots\ldots(s^2+\omega_{\beta m}^2)}{s(s^2+\omega_{\alpha 1}^2)(s^2+\omega_{\alpha 2}^2)(s^2+\omega_{\alpha 3}^2)\ldots\ldots(s^2+\omega_{\alpha m-1}^2)} \quad (4\text{-}11)$$

The procedure for synthesizing the set of parameters given in (4-10) and (4-11) is as follows:

(a) Decompose the composite parameters $-y_{12}$ and y_{22} given in (4-10) and (4-11), in accordance with (4-9). The component parameters of $-y_{12}$, $y_{12}^{(k)}$ can be defined as

$$-y_{12}^{(1)} = \frac{P_1(s)}{Q(s)} ,$$

$$-y_{12}^{(2)} = \frac{P_2(s)}{Q(s)} , \quad (4\text{-}12)$$

$$\vdots$$

$$-y_{12}^{(n)} = \frac{P_n(s)}{Q(s)}$$

In order to realize a set of parameters with a simple LC ladder network, it is necessary that transmission zeros of the component function lie only on the imaginary axis of the s plane including s=0 and s=oo. It can be shown that this decomposition may be accomplished even when the transmission zeros do not lie on the imaginary axis, since the positiveness of coefficients a_k in (4-10) is assured if P(s) is a Hurwitz polynomial in the variable s^2(4).

For the decomposition of y_{22}, the component functions, $y_{22}^{(k)}$, can be defined as

$$
\begin{aligned}
y_{22}^{(1)} &= M_1 \frac{A_1(s)}{Q(s)} \quad , \\
y_{22}^{(2)} &= M_2 \frac{A_2(s)}{Q(s)} \quad , \\
&\vdots \\
y_{22}^{(n)} &= M_n \frac{A_n(s)}{Q(s)}
\end{aligned}
\qquad (4\text{-}13)
$$

where M_k must satisfy the relationship as

$$M_1 + M_2 + \ldots\ldots + M_n = M \qquad (4\text{-}14)$$

(b) Synthesize the set of component functions, $- y_{12}^{(1)}$ and $y_{22}^{(1)}$ to obtain network A in Fig. 30. This step requires simultaneous zero shifting and zero retaining procedures discussed in Chapter II and III. This synthesized network has the same form as the one synthesized by the modified zero shifting synthesis method.

(c) Repeat step 2 for the other sets of component functions $- y_{12}^{(2)}$, $y_{22}^{(2)}$; $- y_{12}^{(3)}$, $y_{22}^{(3)}$;.....; $- y_{12}^{(n)}$, $y_{22}^{(n)}$ and connect each component networks in parallel as in Fig. 30.

(d) Determine transfer ratio K_i from each synthesized network. Each component network has the short-circuit driving function $y_{22}^{(k)}$ and transfer function $- K_i\, y_{12}^{(k)}$. Determine, next, the value of K by assigning

$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2} + \ldots\ldots\ldots\frac{1}{K_n} \tag{4-15}$$

(e) Adjust the respective admittance levels of component networks by multiplying by K/K_i.

As an illustration, consider the problem of synthesizing

$$y_{22}(s) = \frac{(s^2+1)(s^2+5)}{s\,(s^2+2)} \tag{4-16}$$

$$-\,y_{12}(s) = \frac{9 + 5s^2 + s^4}{s(s^2+2)} \tag{4-17}$$

with an LC network.

Decomposition of $-\,y_{12}(s)$ in (4-17), according to (4-12), yields

$$-\,y_{12}(s) = \frac{s^2(s^2+1) + 4(s^2+1) + 5}{s\,(s^2+2)}$$

$$= \frac{(s^2+1)(s^2+4)}{s(s^2+2)} + \frac{5}{s(s^2+2)} \tag{4-18}$$

Then,

$$-\,y_{12}^{(1)}(s) = \frac{(s^2+1)(s^2+4)}{s(s^2+2)} \tag{4-19}$$

$$-\,y_{12}^{(2)}(s) = \frac{5}{s(s^2+2)} \tag{4-20}$$

The component function $-y_{12}^{(1)}$ has a zero at $s=j1$, which is the same as one of $y_{22}(s)$, and the other zero at $s=j2$, while $-\,y_{12}^{(2)}$ has all its zeros at infinity. To synthesize $y_{22}(s)$ with the transmission zeros of $-y_{12}^{(1)}(s)$, the procedure given in Chapter II can be utilized but on an admittance basis. The network arrangement for component network A is shown in Fig. 31.

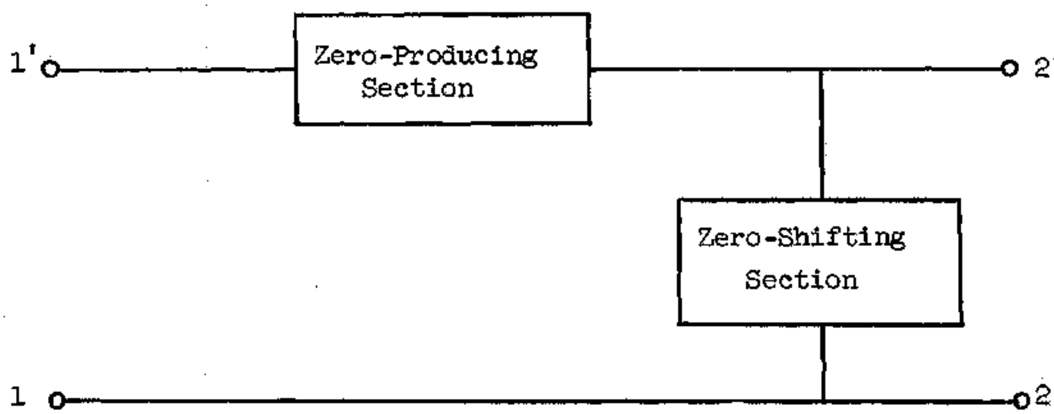

Fig. 31 Network Arrangement for Component Network A

The zero shifting section in Fig. 31 may be realized as follows: Define component functions of the zero shifting section as

$$y_o(s) = \frac{d_o}{s} \tag{4-21}$$

and

$$y_1(s) = \frac{d_1 \ s}{s^2 + 2} \tag{4-22}$$

By analogy with (2-70), the matrix equation needed to determine d_o and d_1 in (4-21) and (4-22) becomes in this case

$$\begin{pmatrix} -1 & \frac{1}{2-1} \\ -\frac{1}{2} & \frac{2}{2-4} \end{pmatrix} \cdot \begin{pmatrix} d_o \\ d_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{4} \end{pmatrix} \tag{4-23}$$

Solving (4-23) for d_o and d_1 gives

$$d_o = 1/2$$
$$d_1 = 1/2 \tag{4-24}$$

The remainder function $y_{22}'(s)$ is

$$\begin{aligned} y_{22}'(s) &= y_{22}(s) - y_o(s) - y_1(s) \\ &= \frac{(s^2+1)(s^2+4)}{s(s^2+2)} \end{aligned} \tag{4-25}$$

To synthesize the zero producing section in Fig. 31 a partial expansion of $z_{22}'(s) = 1/y_{22}'(s)$ is required as

$$\begin{aligned} z_{22}'(s) &= \frac{s(s^2+2)}{(s^2+1)(s^2+4)} \\ &= \frac{1/3\ s}{s^2+1} + \frac{2/3\ s}{s^2+4} \end{aligned} \tag{4-26}$$

which leads to the ladder network shown in Fig. 32. By considering the asymptotic behavior,

$$K_1 = 1 \tag{4-27}$$

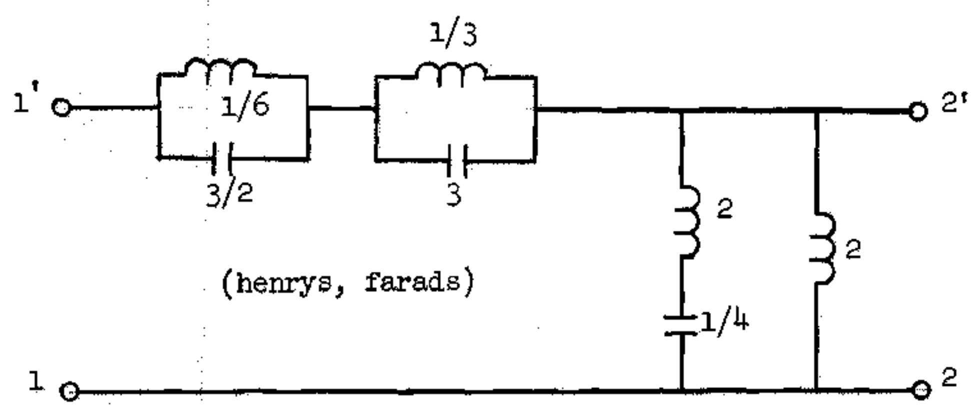

Fig. 32 Network A

Since all the zeros of $-y_{12}^{(2)}$ are at infinity, the appropriate continued fraction expansion of $y_{22}(s)$ takes the form

$$y_{22}(s) = s + \cfrac{1}{1/4\ s + \cfrac{1}{16/3\ s + \cfrac{1}{3/20\ s}}} \tag{4-28}$$

and the pertinent ladder network is shown in Fig. 33. Consideration of the asymptotic behavior gives

$$K_2 = 1 \tag{4-29}$$

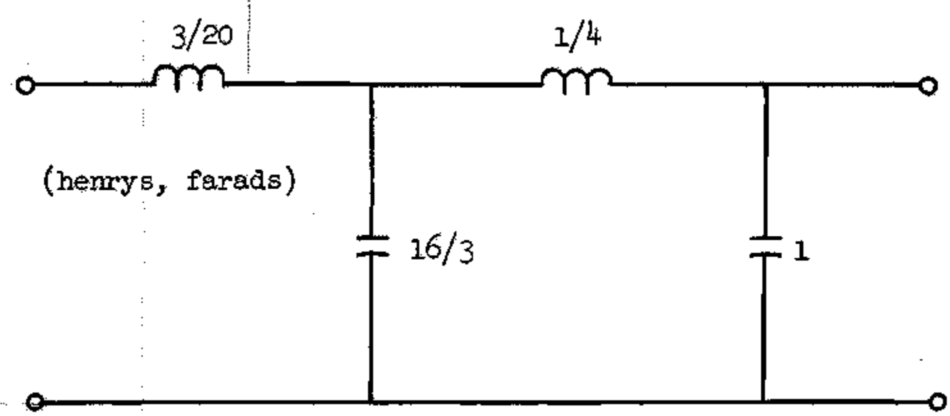

Fig. 33 Network B

Using (4-15) to determine the value of K yields

$$K = 1/2 \tag{4-30}$$

Therefore, the adjustments of admittance levels are required as

for network A, $K/K_1 = 1/2$,

for network B, $K/K_2 = 1/2$.

After adjusting the admittance levels, the resulting parallel ladders are shown in Fig. 34.

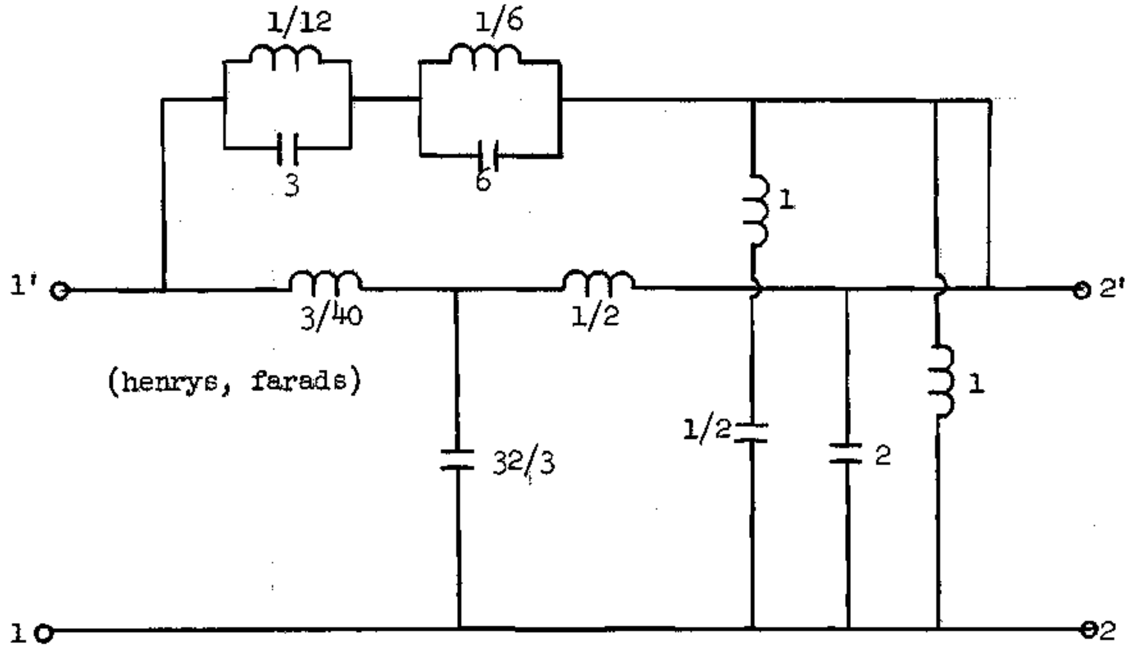

Fig. 34 Network for Synthesizing a Set of Admittance Parameters in (4-16) and (4-17).

CHAPTER V

CONCLUSIONS

This paper has given the methods for simultaneous zero shifting and zero retaining. This study also shows that the success (or failure) of these methods depends upon the proper choice of control frequencies. Simple rules were given for determining proper control frequencies (or component functions).

The proofs of the conditions for realizability by these methods were given in Art. 2.3. It should be pointed out that in order to extend these methods to the RC and RL cases, frequency transformations are required.

The methods can be utilized in the synthesis of transfer functions. The modified zero shifting synthesis or parallel-ladder synthesis given in Chapter IV is one of the applications of these methods, but they can also be applied to other synthesis methods.

As mentioned in Chapter III, the method for shifting i zeros to desired locations and leaving the others the same in a n-z reactance function can always be realized with n control frequencies, if negative elements are allowed. The method with n control frequencies for the proposed zero shifting requires some limitations on the frequency values of new zeros as in Table 3.1. With n+1 control frequencies, those limitations can be dispensed with, but more computations are needed.

An unsuccessful attempt was made to extend the methods to include the case of shifting zeros to proper locations and creating poles at desired positions simultaneously. An interesting fact is that if one pole is created along the positive frequency axis, then an adjacent zero is shifted toward that pole and the other zero is created between the pole to be created and the original pole. It can be shown that if creation of poles is allowed for zero shifting, then the limitations on the value of new zeros given in Table 3 can be eliminated.

Instead of considering reactance parameters, it is often convenient to use susceptance parameters. All the procedures in Chapter II and III are applicable to the susceptance parameters as well.

It would be desirable to extend the conditions given in Table 1 and Table 3 to include zero shifting for n-z functions. It would also be desirable to extend these conditions to the case of zero creating and zero shifting.

APPENDIX

CLASSIFICATION OF COMPONENT NETWORKS

The component networks of zero shifting sections can take any combination of Type I, II and III sections defined in Art. 2.1.

On the other hand, based on the poles and zeros of a reactance function, an orderly arrangement of minimum element networks of increasing complexity is possible. All the possible component networks are defined by Table A.1 or extensions thereof.

Table A.1 Classification of Reactive Networks

Type I	Type II	Type III	Type I II III
X_1	X_2	X_3	X_{123}
X_{12}	X_{23}	X_{33}	X_{1233}
X_{13}	X_{233}	X_{333}	X_{12333}
X_{133}	X_{2333}	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In Table A.1, such symbolisms as X_{12}, X_{13}, etc. are used to indicate, respectively, reactance functions obtained by combining Type I and II sections, obtained by combining Type I and III sections, etc.

The behaviors of these component functions along the positive frequency axis and their reactive and susceptive patterns are represented in Fig. A.1.

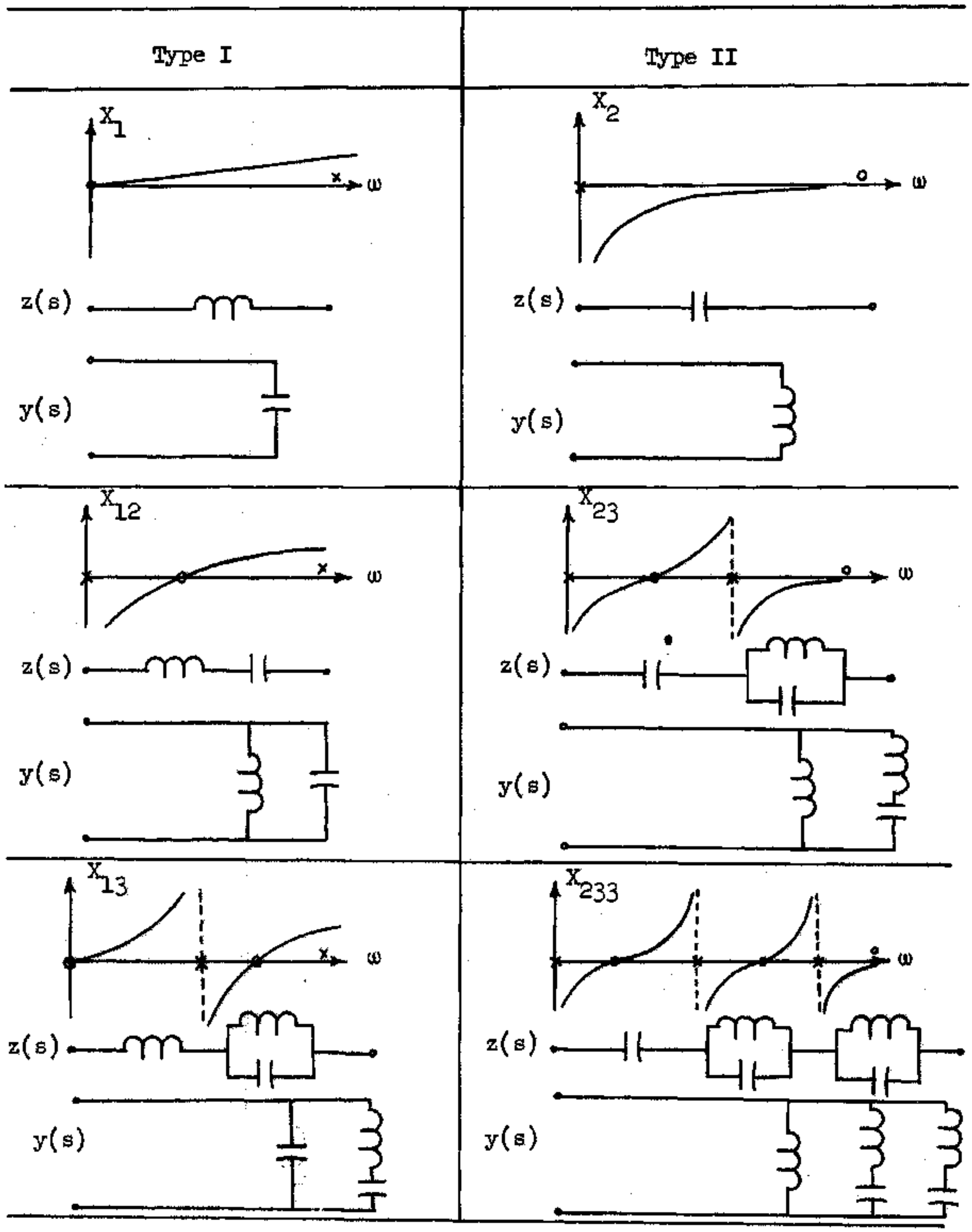

Fig. A.1 (a) Behaviors of Component Networks

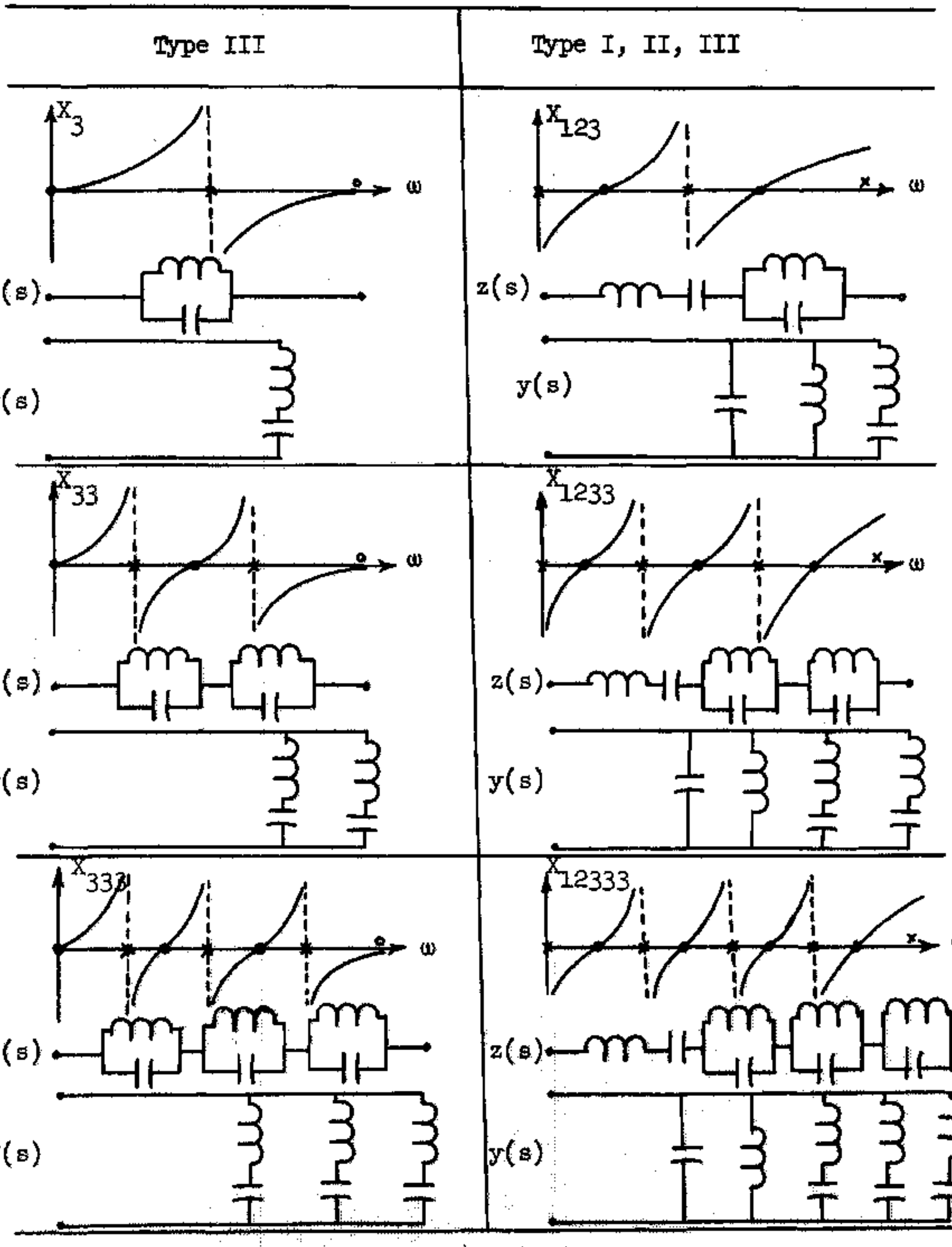

Fig. A.1 (b) Behaviors of Component Functions

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